INDIAN FOREST SERVICE (MAIN) EXAM-2022

JHGT-U-MTH

MATHEMATICS

Paper – I

Time Allowed : ***Three*** *Hours*

Maximum Marks : ***200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

There are ***EIGHT*** *questions in all, out of which* ***FIVE*** *are to be attempted.*

Questions no. ***1*** *and* ***5*** *are* ***compulsory****. Out of the remaining* ***SIX*** *questions,* ***THREE*** *are to be attempted selecting at least* ***ONE*** *question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

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Answers must be written in ***ENGLISH*** *only.*

SECTION A

Q1. (a) Let U and W be subspaces of a vector space V and $x, y \in V$. Then prove that $x + U \subseteq y + W$ iff $U \subseteq W$ and $x - y \in W$. *8*

(b) Let $v_1 = (1, 1, -1)$, $v_2 = (4, 1, 1)$, $v_3 = (1, -1, 2)$ be a basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that $Tv_1 = (1, 0)$, $Tv_2 = (0, 1)$, $Tv_3 = (1, 1)$. Describe the linear transformation T. *8*

(c) Evaluate $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$. *8*

(d) If $x + y + z = u$, $y + z = uv$, $z = uvw$, then determine $\frac{\partial(x, y, z)}{\partial(u, v, w)}$. *8*

(e) A variable plane is at a constant distance of 6 units from the origin and meets the axes in A : (a, 0, 0), B : (0, b, 0) and C : (0, 0, c). Find the locus of the centroid of the triangle ABC. *8*

Q2. (a) Are the matrices $A = \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ similar ? Justify your answer. *10*

(b) Using Lagrange's undetermined multipliers method, find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. *15*

(c) Obtain the coordinates of the points where the shortest distance line between the straight lines $\frac{x-3}{-1} = \frac{y-2}{2} = \frac{z-2}{-1}$; $\frac{x-2}{2} = \frac{y+3}{3} = \frac{z+2}{2}$ meets them. Also find the magnitude of the shortest distance and the equation of the shortest distance line between the straight lines mentioned above *15*

Q3. (a) Reduce the following quadratic form over the real field $\mathbb{R}$ to orthogonal form : *10*

$$q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4xz$$

(b) Find the centre of mass of a solid bounded below by $x^2 + y^2 \leq 4$, $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$. Take the density of the solid as uniform. *15*

(c) Find the equation of the cylinder whose generators intersect the curve, $2x^2 + 3y^2 = 4z$, $x - y + 2z = 3$ and are parallel to the line $3x = -2y = 4z$. *15*

Q4. (a) Show that the improper integral

$$\int_0^{\infty} x^{m-1} e^{-x} \, dx$$

is convergent if $m > 0$. *10*

(b) Let V be the complex vector space of 3×3 skew-symmetric matrices with complex entries i.e. $V = \{A \in M_{3 \times 3}(\mathbb{C}) \mid A^t = -A\}$.

$$\text{Let } B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define a linear transformation $T : V \to V$ by $T(A) = BA - AB$. Find the eigenvalues and eigenvectors of T. *15*

(c) Reduce the equation,

$$3x^2 + 6yz - y^2 - z^2 - 6x + 6y + 2z + 2 = 0$$

to a canonical form and mention the name of the surface it represents. *15*

SECTION B

Q5. (a) Find the general solution of the differential equation :

$$p^2 \cos^2 y + p \sin x \cos x \cos y - \sin y \cos^2 x = 0, \text{ where } p \equiv \frac{dy}{dx}.$$ 8

(b) Solve the differential equation :

$$(D^2 - 1)y = e^x (1 + x^2), \text{ where } D \equiv \frac{d}{dx}.$$ 8

(c) Three forces P, Q and R act along the sides BC, CA and AB of Δ ABC in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$$\frac{1 + \cos \alpha}{P} + \frac{1 + \cos \beta}{Q} + \frac{1 + \cos \gamma}{R} = 0,$$

where α, β, γ are the interior angles subtended at A, B, C respectively. 8

(d) A person is drawing water from a well with a light bucket which leaks uniformly. The bucket weighs 50 kg when it is full. When it arrives at the top, half of the water remains inside. If the depth of the water level in the well from the top is 30 m, then find the work done in raising the bucket to the top from the water level. 8

(e) Determine constants a, b, c so that the directional derivative of $\phi(x, y, z) = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 88 in a direction parallel to z-axis. 8

Q6. (a) Solve the differential equation :

$$4(xp^2 + yp) = y^4, \text{ where } p \equiv \frac{dy}{dx}.$$ 10

(b) A particle of mass m, which is attached to one end of a light string whose other end is fixed at a point O, describes a circular motion in a horizontal plane about the vertical axis through O. Prove that the particle moves in a conical pendulum only if $g < l\omega^2$, where l is the length of the string and ω being angular velocity.

Further, a particle of mass m is attached to the middle of a light string of length $2l$, one end of which is fastened to a fixed point and the other end to a smooth ring of mass M which slides on a smooth vertical rod. If the particle describes a horizontal circle with uniform angular velocity ω about the rod, then prove that the inclination of both portions of the string to the vertical is

$$\cos^{-1}\left\{\frac{(m + 2M)g}{m\, l\, \omega^2}\right\}.$$ *15*

(c) Given that C is a curve of the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 2$ and C is described counterclockwise.

Verify Stokes' theorem for the line integral $\int_C - y^3dx + x^3dy - z^3dz$. *15*

Q7. (a) Derive vector identity for divergence of cross product of two vector point functions. Given a relation between linear and angular velocity as $\vec{v} = \vec{\omega} \times \vec{r}$.

If $\vec{\omega}$ is constant, then show that

(i) curl $\vec{v} = 2\vec{\omega}$

(ii) div $\vec{v} = 0$. *10*

(b) Given that $y_1 = x^2$ is a solution of the differential equation,

$$x^3 \frac{d^2y}{dx^2} - (x^2 + 3x)\frac{dy}{dx} + 6y = 0,$$

find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. *15*

(c) PR and QR are two equal heavy strings tied together at R and carrying a weight W at R. P and Q are two points in the same horizontal line and 2a is the distance between them. l is the length of each string and h is the depth of R below PQ. Prove that

(i) $l^2 - h^2 = 2c^2 \left(\cosh \frac{a}{c} - 1\right)$,

(ii) Tension at P or Q $= \frac{1}{2h} \{lW + (l^2 + h^2)w\}$,

where c is the parameter of the catenary and w is the line density of the string. *15*

Q8. (a) A bucket is in the form of a frustum of a cone and is filled with water of density ρ. If the bottom and top ends of the bucket have radii a and b respectively and h is the height of the bucket, then find the resultant vertical thrust on the curved surface of the bucket. Is that thrust equal to $\frac{1}{3} \pi \rho g h (b - a)(b + 2a)$? *10*

(b) Solve the differential equation :

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 15x^4 + 8x^3.$$ *15*

(c) If a curve in a space is represented by $\vec{r} = \vec{r}(t)$, then derive expressions of its torsion and curvature in terms of $\dot{\vec{r}}$, $\ddot{\vec{r}}$ and $\dddot{\vec{r}}$. Find the curvature and torsion of the curve given by $\vec{r} = (at - a \sin t,\ a - a \cos t,\ bt)$. *15*

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MATHEMATICS

Paper – II

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SECTION A

Q1. (a) Let F be a finite field of characteristic p, where p is a prime. Then show that there is an injective homomorphism from $\mathbb{Z}_p$ (group of integers modulo p) to F. Also show that number of elements in F is p^n, for some positive integer n. *8*

(b) Let $\mathbb{R}$ denote the set of real numbers and Q denote the set of rational numbers. If $x \in \mathbb{R}$, $x > 0$ and $y \in \mathbb{R}$, then show that there exists a positive integer n such that $nx > y$. Use it to show that if $x < y$, then there exists $p \in Q$ such that $x < p < y$. *8*

(c) Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Then show that f is Riemann integrable on [a, b]. *8*

(d) Prove that the linear programming problem

Maximize

$$z = 3x_1 + 2x_2$$

subject to the constraints :

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

does not admit an optimum basic feasible solution. *8*

(e) Compute the integral

$$\oint_C \frac{1 + 2z + z^2}{(z-1)^2 (z+2)} dz$$

where C is $|z| = 3$. *8*

Q2. (a) Find all the Sylow p-subgroups of S_4 and show that none of them is normal. *10*

(b) Suppose $\{f_n\}$ is a sequence of functions defined on $[a, b]$ and $\lim_{n \to \infty} f_n(x) = f(x)$, and $x \in [a, b]$. Put $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$.

Then show that

(i) f_n converges to f uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$. *7*

(ii) If $|f_n(x)| \leq M_n$, $(x \in [a, b], n = 1, 2, \ldots)$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ if $\sum_{n=1}^{\infty} M_n$ converges. *8*

(c) Find a bilinear transformation $w = f(z)$ which maps the upper half plane $\text{Im}(z) \geq 0$ onto the unit disk $|w| \leq 1$. *15*

Q3. (a) (i) Prove that every bounded and monotonically increasing sequence is convergent and converges to lub (least upper bound) of the sequence. *5*

(ii) If $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, $\forall\, n \in \mathbb{N}$, then using Cauchy criterion for convergence of the sequence, show that $\{a_n\}$ is not convergent. *5*

(b) (i) Let P be a Sylow p-subgroup of a group G and H is any p-subgroup of G such that HP = PH. Then show that $H \subseteq P$. *7*

(ii) Show that every group of order 15 is cyclic. *8*

(c) Employ duality to solve the following linear programming problem : *15*

Maximize

$$z = 2x_1 + x_2$$

subject to the constraints :

$$x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 - 2x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Q4. (a) (i) Find an upper bound for the absolute value of the integral $I = \int_C e^z \, dz$, where C is the line segment joining the points (0, 0) and (1, 3). *8*

(ii) Find the length of the curve C defined by

$z(t) = (1 - 2i)t^3, \ -1 \le t \le 1.$ *7*

(b) Prove that R[x] is a principal ideal domain if and only if R is a field. *10*

(c) Find the initial basic feasible solution to the following transportation problem by the North-West corner rule and then optimize it. *15*

	To			Availability
From	7	3	4	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

SECTION B

Q5. (a) Equation of any cone with vertex at the point (a, b, c) is of the form $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$. Find the partial differential equation of the cone. *8*

(b) Given f(1) = 4, f(2) = 5, f(7) = 5 and f(8) = 4. Find the value of f(6) and also the value of x for which f(x) is maximum or minimum. *8*

(c) (i) If $x = 0{\cdot}101010101E0001010$ and $y = 0{\cdot}100010110E0000110$, then find $x - y$. *4*

(ii) Draw the map of the Boolean function $F = x'yz + xy'z' + xyz + xyz'$. Also simplify the function. *4*

(d) A rod of length 2a revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α. Prove that the direction of reaction at the hinge makes with the vertical, an angle $\tan^{-1}\left[\frac{3}{4}\tan\alpha\right]$. *8*

(e) Verify that the equation $yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0$ is integrable and find its solution. *8*

Q6. (a) Find the system of equations for obtaining the general equation of surfaces orthogonal to the family given by

$$x(x^2 + y^2 + z^2) = Cy^2,$$

where C is a parameter. *10*

(b) Write down an algorithm for Simpson's $\frac{1}{3}$ rule. Hence, compute $\int_0^1 x^2(1-x)\,dx$ correct up to three decimal places with step size $h = 0{\cdot}1$ and compare the result with its exact value. *6+9*

(c) If the velocity of an incompressible fluid at the point (x, y, z) is given by (– Ay, Ax, 0), then prove that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of the fluid flow. *15*

Q7. (a) Solve the following system of equations by Gauss-Jordan method : *15*

$$2x + y - 3z = 11$$
$$4x - 2y + 3z = 8$$
$$-2x + 2y - z = -6$$

(b) Verify that $w = i\,k \log\left(\dfrac{z - ia}{z + ia}\right)$ is the complex potential of a steady fluid flow about a circular cylinder, the plane y = 0 being a rigid boundary.

Further show that the fluid exerts a downward force of magnitude $\left(\dfrac{\pi \rho k^2}{2a}\right)$ per unit length on the cylinder, where ρ is the fluid density. *15*

(c) Find the solution of the partial differential equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y); \;\; p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}$$

which passes through the x-axis, using Cauchy's method of characteristics. *10*

Q8. (a) A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is $\phi(r)$ at a distance r from the origin. By employing the principle of energy and least action, show that the path is given by the following differential equation :

$$c^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = r^4\,[h - \phi(r)],$$

where c is a constant. *15*

(b) Find the real root of the equation $e^x - 3x = 0$, by Newton–Raphson method, correct up to four decimal places. *10*

(c) Find a complete integral of the partial differential equation

$$(p^2 + q^2)x = pz; \; p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}$$

using Charpit's method and hence deduce the solution which passes through the curve $x = 0$, $z^2 = 4y$. *15*