Indian Forest Service
(Main) Exam, 2020

JKLO-U-MTH

MATHEMATICS

Paper – I

*Time Allowed : **Three** Hours*	*Maximum Marks : **200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

*Answers must be written in **ENGLISH** only.*

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) If A is a skew-symmetric matrix and I + A be a non-singular matrix, then show that $(I - A)(I + A)^{-1}$ is orthogonal. *8*

(b) By applying elementary row operations on the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix},$$

reduce it to a row-reduced echelon matrix. Hence find the rank of A. *8*

(c) Given that $f(x + y) = f(x) f(y)$, $f(0) \neq 0$, for all real x, y and $f'(0) = 2$. Show that for all real x, $f'(x) = 2 f(x)$. Hence find f(x). *8*

(d) Find the Taylor's series expansion for the function

$$f(x) = \log(1 + x), \quad -1 < x < \infty,$$

about x = 2 with Lagrange's form of remainder after 3-terms. *8*

(e) If the straight lines, joining the origin to the points of intersection of the curve $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$ and the straight line $2x + 3y + k = 0$, are at right angles, then show that $6k^2 + 5k + 52 = 0$. *8*

Q2. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x, -3y, x + y)$, and $B_1 = \{(-1, 2, 0), (0, 1, -1), (3, 1, 2)\}$ be a basis of $\mathbb{R}^3$. Find the matrix representation of T relative to the basis B_1. *10*

(b) Using Lagrange's multiplier, show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube. *15*

(c) Prove that the angle between two straight lines whose direction cosines are given by $l + m + n = 0$ and $fmn + gnl + hlm = 0$ is $\frac{\pi}{3}$, if $\frac{1}{f} + \frac{1}{g} + \frac{1}{h} = 0$. *15*

Q3. (a) Find the asymptotes of the curve $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$. *10*

(b) When is a matrix A said to be similar to another matrix B ?

Prove that

(i) if A is similar to B, then B is similar to A.

(ii) two similar matrices have the same eigenvalues.

Further, by choosing appropriately the matrices A and B, show that the converse of (ii) above may not be true. *15*

(c) A point P moves on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, which is fixed. The plane through P and perpendicular to OP meets the axes in A, B, C respectively. The planes through A, B, C parallel to yz, zx and xy planes respectively intersect at Q. Prove that the locus of Q is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}.$$ *15*

Q4. (a) Let P be the vertex of the enveloping cone of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. If the section of this cone made by the plane z = 0 is a rectangular hyperbola, then find the locus of P. *10*

(b) (i) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, hence find its inverse. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$ as a linear polynomial in A.

(ii) Express the vector (1, 2, 5) as a linear combination of the vectors (1, 1, 1), (2, 1, 2) and (3, 2, 3), if possible. Justify your answer. *9+6=15*

(c) (i) Evaluate :

$$\lim_{x \to 1} (x - 1) \tan \frac{\pi x}{2}.$$

(ii) Evaluate the following integral :

$$\int_{-\infty}^{\infty} x e^{-x^2} \, dx$$

6+9=15

SECTION B

Q5. (a) Solve the initial value problem :

$$(2x^2 + y)\,dx + (x^2y - x)\,dy = 0, \;\; y(1) = 2.$$ *8*

(b) Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 16x - 12e^{2x}.$$ *8*

(c) If the radial and transverse velocities of a particle are proportional to each other, then prove that the path is an equiangular spiral. Further, if radial acceleration is proportional to transverse acceleration, then show that the velocity of the particle varies as some power of the radius vector. *8*

(d) A cylinder of radius 'r', whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length l and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. Prove that the reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $\frac{W}{2}$. Also, prove that the ratio of radius of the cylinder to the length of the beam is $5 + \sqrt{5} : 4\sqrt{2}$. *8*

(e) Prove that for a vector $\vec{a}$,

$$\nabla(\vec{a}\cdot\vec{r}) = \vec{a}\,; \text{where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\; r = |\vec{r}|.$$

Is there any restriction on $\vec{a}$?

Further, show that

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3}$$

Give an example to verify the above. *8*

Q6. (a) Find one solution of the differential equation

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

by inspection and using that solution determine the other linearly independent solution of the given equation. Obtain the general solution of the given differential equation. *10*

(b) A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant 'b' from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre ? *15*

(c) A tangent is drawn to a given curve at some point of contact. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{[25\,\kappa^2\,\tau^2\,(1 + 25\,\kappa^2) + \{\kappa + 5\,\frac{d\kappa}{ds} + 25\,\kappa^3\}]^{1/2}}{(1 + 25\,\kappa^2)^{3/2}}.$$

Find the curvature and torsion of the curve $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$. *15*

Q7. (a) Derive intrinsic equation

$$x = c \log(\sec\psi + \tan\psi)$$

of the common catenary, where symbols have usual meanings.

Prove that the length of an endless chain, which will hang over a circular pulley of radius 'a' so as to be in contact with $\frac{2}{3}$ of the circumference of the pulley, is

$$a\left\{\frac{4\pi}{3} + \frac{3}{\log(2 + \sqrt{3})}\right\}.$$

10

(b) Solve the differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$$ 15

(c) Given a portion of a circular disc of radius 7 units and of height 1·5 units such that $x, y, z \geq 0$.

Verify Gauss Divergence Theorem for the vector field

$$\vec{f} = (z,\ x,\ 3y^2 z)$$

over the surface of the above mentioned circular disc. 15

Q8. (a) Derive expression of ∇f in terms of spherical coordinates.

Prove that

$$\nabla^2 (fg) = f\,\nabla^2 g + 2\,\nabla f \,.\, \nabla g + g\,\nabla^2 f$$

for any two vector point functions $f(r, \theta, \phi)$ and $g(r, \theta, \phi)$.

Construct one example in three dimensions to verify this identity. 10

(b) Reduce the differential equation

$$xp^2 - 2yp + x + 2y = 0, \quad \left(p = \frac{dy}{dx}\right),$$

to Clairaut's form and obtain its complete primitive. Also, determine a singular solution of the given differential equation. 15

(c) A sphere of radius 'a', and having density half of that of water, is completely immersed at the bottom of a circular cylinder of radius 'b', which is filled with water to depth 'd'. The sphere is set free and takes up its position of equilibrium. Show that the loss of potential energy this way is

$$W\left(d - \frac{11}{8}a - \frac{a^3}{3b^2}\right),$$

where W is the weight of the sphere. 15

Indian Forest Service (Main) Exam, 2020

JKLO–B–MTH

MATHEMATICS

PAPER—II

Time Allowed : Three Hours

Maximum Marks : 200

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions

There are EIGHT questions in all, out of which FIVE are to be attempted.

Question Nos. **1** and **5** are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary and indicate the same clearly.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

Answers must be written in ENGLISH only.

SECTION—A

1. *(a)* Let p be a prime number. Then show that

$$(p-1)!+1 \equiv 0 \bmod (p)$$

Also, find the remainder when $6^{44} \cdot (22)!+3$ is divided by 23. 8

(b) *(i)* If $u=u(y-z,\ z-x,\ x-y)$, then find the value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$.

(ii) If $u(x, y, z)=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$, then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$. 8

(c) Evaluate the integral $\iint_R (x-y)^2 \cos^2(x+y)\,dx\,dy$, where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$. 8

(d) Solve graphically the following LPP :

$$\text{Max} \quad z=5x_1-3x_2$$

subject to

$$3x_1+2x_2 \le 12$$
$$-x_1+x_2 \ge 1$$
$$-x_1+x_2 \le 2$$
$$x_1,\ x_2 \ge 0$$

If the objective function z is changed to Max $z=6x_1+4x_2$, while the constraints remain the same, then comment on the number of solutions. Will $(4, 0)$ be also a solution? 8

(e) Evaluate the integral $\int_C \text{Re}(z^2)\,dz$ from 0 to $2+4i$ along the curve $C: y=x^2$. 8

2. *(a)* Let R be a non-zero commutative ring with unity. Show that M is a maximal ideal in a ring R if and only if R/M is a field. 10

(b) Show that the sequence of functions $\{f_n(x)\}$, where $f_n(x)=nx(1-x)^n$, does not converge uniformly on $[0, 1]$. 15

(c) Using Cauchy theorem and Cauchy integral formula, evaluate the integral

$$\oint_C \frac{e^z}{z^2(z+1)^3}\, dz$$

where C is $|z|=2$. 15

3. (a) Find the extreme values of $f(x, y, z) = 2x+3y+z$ such that $x^2+y^2=5$ and $x+z=1$. 10

(b) Let G be a finite group and let p be a prime. If p^m divides order of G, then show that G has a subgroup of order p^m, where m is a positive integer. 15

(c) Solve the following LPP by simplex method :

$$\text{Max} \quad z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 2x_1 - 2x_2 &\le 1 \\ 2x_1 - 4x_2 &\le 3 \\ 2x_1 + x_2 &\le 2 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Does there exist an alternate optimal solution? If yes, give one and hence find all the optimal solutions. 15

4. (a) Show that the bilinear transformation

$$w = e^{i\theta_0}\left(\frac{z-z_0}{z-\bar{z}_0}\right)$$

z_0 being in the upper half of the z-plane, maps the upper half of the z-plane into the interior of the unit circle in the w-plane. If under this transformation, the point $z=i$ is mapped into $w=0$ while the point at infinity is mapped into $w=-1$, then find this transformation. 10

(b) Let K be a finite field. Show that the number of elements in K is p^n, where p is a prime, which is characteristic of K and $n \ge 1$ is an integer. Also, prove that $\dfrac{\mathbb{Z}_3[X]}{(X^2+1)}$ is a field. How many elements does this field have? 15

(c) Find the minimum transportation cost using Vogel's approximation method for the following transportation problem :

Destinations

		D_1	D_2	D_3	D_4	*Availability*
	S_1	9	16	15	9	15
Sources	S_2	2	1	3	5	25
	S_3	6	4	7	3	20
Demand		10	15	25	10	

15

SECTION—B

5. (a) Construct a partial differential equation of all surfaces of revolution having the z-axis as the axis of rotation. 8

(b) Using Newton-Raphson method, find the value of $(37)^{1/3}$, correct to four decimal places. 8

(c) Answer the following questions :

(i) Convert $(14231)_8$ into an equivalent binary number and then find the equivalent decimal number.

(ii) Convert $(43503)_{10}$ into an equivalent binary number and then find the equivalent hexadecimal number. 8

(d) Find the condition on a, b, c (real numbers) such that the dynamical system with equations $\dot{p} = aq - q^2$, $\dot{q} = bp + cq$ is Hamiltonian. Compute also the Hamiltonian of the system. 8

(e) Find the general solution of the partial differential equation

$$p \tan x + q \tan y = \tan z$$

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$. 8

6. (a) Find the general solution and singular solution of the partial differential equation

$$6yz - 6pxy - 3qy^2 + pq = 0$$

10+5=15

(b) Find the Lagrange interpolating polynomial that fits the following data values :

x :	-1	2	3	5
$f(x)$:	-1	10	25	60

Also, interpolate at $x = 2{\cdot}5$, correct to three decimal places. 15

(c) In a fluid flow, the velocity vector is given by $\vec{V} = 2x\vec{i} + 3y\vec{j} - 5z\vec{k}$. Determine the equation of the streamline passing through a point $A = (4, 8, 1)$. 10

7. (a) Write down the algorithm and flowchart for solving numerically the differential equation $\frac{dy}{dx} = f(x, y) = 1 + x\cos y$ with initial condition : at $x = x_0$, $y = y_0$ and step length h by Euler's method up to $x = x_n = x_0 + nh$. 7+8=15

(b) In a two-dimensional fluid flow, the velocity components are given by $u = x - ay$ and $v = -ax - y$, where a is constant. Show that the velocity potential exists for this flow and determine the appropriate velocity potential. Also, determine the corresponding stream function that would represent the flow. 15

(c) Find the solution of the following differential equation :

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} = ye^x$$ 10

8. (a) A particle is attracted to a center by a force which varies inversely as the cube of its distance from the center. Identify the generalized coordinates and write down the Lagrangian of the system. Derive then the equations of motion and solve them for the orbits. Discuss how the nature of orbits depends on the parameters of the system. 20

(b) Evaluate the integral $\int_0^2 \frac{x}{1+x^3}\,dx$, using trapezoidal rule with $h = \frac{1}{4}$, correct to three decimal places. (h is the length of subinterval) 10

(c) Solve the following system of linear equations using Gaussian elimination method :

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= -2 \\ 6x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$ 10

★ ★ ★