C–GEQ–O–NBUA

MATHEMATICS

PAPER—I

Time Allowed : Three Hours	Maximum Marks : 200

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions

There are EIGHT questions in all, out of which FIVE are to be attempted.

Question Nos. **1** and **5** are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in ENGLISH only.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary and indicate the same clearly.

SECTION—A

1. *(a)* Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$. 8

(b) Let G be the linear operator on $\mathbb{R}^3$ defined by

$$G(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix representation of G relative to the basis

$$S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$ 8

(c) Let $f(x)$ be a real-valued function defined on the interval $(-5, 5)$ such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\, dt$ for all $x \in (-5, 5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$. 8

(d) For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t} dt$. Evaluate $f(e) + f\left(\frac{1}{e}\right)$. 8

(e) The tangent at $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse. 8

2. *(a)* Suppose U and W are distinct four-dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of $U \cap W$. 10

(b) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution : 10

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

(c) Consider the three-dimensional region R bounded by $x + y + z = 1$, $y = 0$, $z = 0$. Evaluate $\iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$. 10

(d) Find the area enclosed by the curve in which the plane $z = 2$ cuts the ellipsoid

$$\frac{x^2}{25} + y^2 + \frac{z^2}{5} = 1$$ 10

3. (a) Find the minimal polynomial of the matrix $A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$. 10

(b) If $\sqrt{x+y} + \sqrt{y-x} = c$, find $\dfrac{d^2y}{dx^2}$. 10

(c) A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensions of the box so that the total surface is a minimum. 10

(d) Find the equation of the plane containing the straight line $y+z=1$, $x=0$ and parallel to the straight line $x-z=1$, $y=0$. 10

4. (a) Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$. 10

(b) Find the locus of the variable straight line that always intersects $x=1$, $y=0$; $y=1$, $z=0$; $z=1$, $x=0$. 10

(c) Find the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$. 10

(d) Evaluate $\lim_{x \to 0} \left(\dfrac{2+\cos x}{x^3 \sin x} - \dfrac{3}{x^4} \right)$. 10

SECTION—B

5. (a) Reduce the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0$, $p = \dfrac{dy}{dx}$ to Clairaut's form. Hence, find the singular solution of the equation. 8

(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point. 8

(c) Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$. 8

(d) A cylindrical vessel on a horizontal circular base of radius a is filled with a liquid of density w with a height h. If a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, determine the increase of the whole pressure on the curved surface. 8

(e) Solve the differential equation $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. 8

6\. *(a)* Solve $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^3\sin x^2$ by changing the independent variable. 10

(b) The forces P, Q and R act along three straight lines $y=b$, $z=-c$; $z=c$, $x=-a$ and $x=a$, $y=-b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action. 10

(c) Solve $(D^4+D^2+1)y=e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right)$, where $D\equiv\frac{d}{dx}$. 10

(d) Examine if the vector field defined by $\vec{F}=2xyz^3\hat{i}+x^2z^3\hat{j}+3x^2yz^2\hat{k}$ is irrotational. If so, find the scalar potential ϕ such that $\vec{F}=\text{grad}\,\phi$. 10

7\. *(a)* Determine the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley. 15

(b) Using divergence theorem, evaluate

$$\iint_S (x^3\,dy\,dz+x^2y\,dz\,dx+x^2z\,dy\,dx)$$

where S is the surface of the sphere $x^2+y^2+z^2=1$. 15

(c) A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t. 10

8\. *(a)* An ellipse is just immersed in water with its major axis vertical. If the centre of pressure coincides with the focus, determine the eccentricity of the ellipse. 15

(b) If $\vec{F}=y\hat{i}+(x-2xz)\hat{j}-xy\hat{k}$, evaluate $\iint_S(\nabla\times\vec{F})\cdot\hat{n}\,dS$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. 10

(c) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path. 15

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I. F.S. EXAM-2015

C-GEQ-O-NBUB

MATHEMATICS
Paper – II

Time Allowed : Three Hours

Maximum Marks : 200

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

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Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) If in a group G there is an element a of order 360, what is the order of a^{220} ? Show that if G is a cyclic group of order n and m divides n, then G has a subgroup of order m. *10*

(b) Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

Suppose $\sum_{n=1}^{\infty} a_{2n} = \frac{9}{8}$ and $\sum_{n=0}^{\infty} a_{2n+1} = \frac{-3}{8}$. What is $\sum_{n=1}^{\infty} a_n$?

Justify your answer. (Majority of marks is for the correct justification). *8*

(c) Let u(x, y) = cos x sinh y. Find the harmonic conjugate v(x, y) of u and express u(x, y) + i v(x, y) as a function of z = x + iy. *12*

(d) Solve graphically :

Maximize $z = 7x + 4y$

subject to $2x + y \leq 2$, $x + 10y \leq 10$ and $x \leq 8$.

(Draw your own graph without graph paper). *10*

Q2. (a) If p is a prime number and e a positive integer, what are the elements 'a' in the ring $\mathbb{Z}_{p^e}$ of integers modulo p^e such that $a^2 = a$? Hence (or otherwise) determine the elements in $\mathbb{Z}_{35}$ such that $a^2 = a$. *14*

(b) Let $X = (a, b]$. Construct a continuous function $f : X \to \mathbb{R}$ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on $[a + \varepsilon, b]$, $a + \varepsilon < b$? Why ? *14*

(c) Evaluate the integral $\int_r \frac{z^2}{(z^2+1)(z-1)^2}\, dz$, where r is the circle $|z| = 2$. *12*

Q3. (a) What is the maximum possible order of a permutation in S_8, the group of permutations on the eight numbers {1, 2, 3, ... , 8} ? Justify your answer. (Majority of marks will be given for the justification). *13*

(b) Let $f_n(x) = \frac{x}{1 + nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f ? Show that for $x \neq 0$, $f_n'(x) \to f'(x)$ but $f_n'(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$. *13*

(c) A manufacturer wants to maximise his daily output of bulbs which are made by two processes P_1 and P_2. If x_1 is the output by process P_1 and x_2 is the output by process P_2, then the total labour hours is given by $2x_1 + 3x_2$ and this cannot exceed 130, the total machine time is given by $3x_1 + 8x_2$ which cannot exceed 300 and the total raw material is given by $4x_1 + 2x_2$ and this cannot exceed 140. What should x_1 and x_2 be so that the total output $x_1 + x_2$ is maximum ? Solve by the simplex method only. *14*

Q4. (a) Compute the double integral which will give the area of the region between the y-axis, the circle $(x - 2)^2 + (y - 4)^2 = z^2$ and the parabola $2y = x^2$. Compute the integral and find the area. *15*

(b) Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$ by using contour integration and the residue theorem. *15*

(c) Solve the following transportation problem : *10*

	D_1	D_2	D_3	Supply
O_1	5	3	6	20
O_2	4	7	9	40
Demand	15	22	23	60

SECTION B

Q5. (a) Store the value of -1 in hexadecimal in a 32-bit computer. *10*

(b) Show that $\sum_{k=1}^{n} l_k(x) = 1$, where $l_k(x)$, k = 1 to n, are Lagrange's fundamental polynomials. *10*

(c) Derive the Hamiltonian and equation of motion for a simple pendulum. *10*

(d) Find the solution of the equation $u_{xx} - 3u_{xy} + u_{yy} = \sin(x - 2y)$. *10*

Q6. (a) Solve the following system of linear equations correct to two places by Gauss-Seidel method :

$$x + 4y + z = -1, \;\; 3x - y + z = 6, \;\; x + y + 2z = 4.$$ *16*

(b) Solve the differential equation $u_x^2 - u_y^2$ by variable separation method. *12*

(c) In a steady fluid flow, the velocity components are $u = 2kx$, $v = 2ky$ and $w = -4kz$. Find the equation of a streamline passing through (1, 0, 1). *12*

Q7. (a) Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; 0 < x < 1, \;\; t > 0$$

subject to the conditions $u(0, t) = u(1, t) = 0$ for $t > 0$ and $u(x, 0) = \sin \pi x, \;\; 0 < x < 1.$ *14*

(b) Find the moment of inertia of a uniform mass M of a square shape with each side a about its one of the diagonals. *12*

(c) Use the classical fourth order Runge-Kutta methods to find solutions at $x = 0{\cdot}1$ and $x = 0{\cdot}2$ of the differential equation $\frac{dy}{dx} = x + y$, $y(0) = 1$ with step size $h = 0{\cdot}1$. *14*

Q8. (a) Write a BASIC program to compute the product of two matrices. *12*

(b) Suppose $\vec{v} = (x - 4y)\hat{i} + (4x - y)\hat{j}$ represents a velocity field of an incompressible and irrotational flow. Find the stream function of the flow. *12*

(c) Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for a string of length l fixed at both ends. The string is given initially a triangular deflection

$$u(x, 0) = \begin{cases} \frac{2}{l} x, & \text{if } 0 < x < \frac{l}{2} \\ \frac{2}{l}(l - x), & \text{if } \frac{l}{2} \le x < l \end{cases}$$

with initial velocity $u_t(x, 0) = 0$. *16*