16 S 2010

005412

B-JGT-K-NBA

MATHEMATICS

Paper I

Time Allowed : Three Hours | *Maximum Marks : 200*

INSTRUCTIONS

Candidates should attempt questions 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks. Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

SECTION A

1. Answer any *five* of the following :

(a) Show that the set

$$P[t] = \{at^2 + bt + c \;/\; a, b, c \in \mathbb{R}\}$$

forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space ? *8*

(b) Determine whether the quadratic form

$$q = x^2 + y^2 + 2xz + 4yz + 3z^2$$

is positive definite. *8*

(c) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$. *8*

(d) Let f be a function defined on $\mathbb{R}$ such that

$$f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}.$$

If f is differentiable at one point of $\mathbb{R}$, then prove that f is differentiable on $\mathbb{R}$. *8*

(e) If a plane cuts the axes in A, B, C and (a, b, c) are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$. *8*

(f) Find the equations of the spheres passing through the circle

$$x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, \quad y = 0$$

and touching the plane $3y + 4z + 5 = 0$. *8*

2. (a) Show that the vectors

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

form a basis for $\mathbb{R}^3$. Find the components of (1, 0, 0) w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. *10*

(b) Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley – Hamilton theorem for this matrix and hence find its inverse. *10*

(c) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. *12*

(d) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$

8

3. (a) Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1 + x^4 \sin^2 x}$$

10

(b) Find the extreme value of xyz if $x + y + z = a$. *10*

(c) Let

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that :

(i) $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

(ii) f is differentiable at (0, 0) *10*

(d) Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$. *10*

4. (a) Prove that the second degree equation

$$x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8x - 19y - 2z - 20 = 0$$

represents a cone whose vertex is (1, –2, 3). *10*

(b) If the feet of three normals drawn from a point P to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ lie in the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, prove that the feet of the other three normals lie in the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + 1 = 0.$$ *10*

(c) If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. *10*

(d) Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}.$$

10

SECTION B

5. Answer any ***five*** of the following :

(a) Show that $\cos(x + y)$ is an integrating factor of

$$y\,dx + [y + \tan(x + y)]\,dy = 0.$$

Hence solve it. *8*

(b) Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$$ *8*

(c) A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability. *8*

(d) A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that,

$$\tan\theta = \tan\theta_1 + \tan\theta_2.$$ *8*

(e) Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which, are in the ratio, 4 : 5. *8*

(f) Find the directional derivation of $\vec{V}^2$, where, $\vec{V} = xy^2\vec{i} + zy^2\vec{j} + xz^2\vec{k}$ at the point (2, 0, 3) in the direction of the outward normal to the surface $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1). . *8*

6. (a) Solve the following differential equation

$$\frac{dy}{dx} = \sin^2(x - y + 6)$$ *8*

(b) Find the general solution of

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 + 1)y = 0$$ *12*

(c) Solve

$$\left(\frac{d}{dx} - 1\right)^2\left(\frac{d^2}{dx^2} + 1\right)^2 y = x + e^x$$ *10*

(d) Solve by the method of variation of parameters the following equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = (x^2 - 1)^2$$ *10*

7. (a) A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is,

$$\frac{1}{2}\left[P.\frac{l}{h} + W.\frac{h^2 + l^2}{2hl}\right].$$ *14*

(b) A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is,

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \frac{2\mu}{R}\right)^{1/2}}\right].$$

13

(c) A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass m slides down its slant face which is inclined at an angle α to the horizontal plane. Prove that the acceleration of the wedge is,

$$\frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.$$

13

8. (a) (i) Show that

$$\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3z^2x\vec{k}$$

is a conservative field. Find its scalar potential and also the work done in moving a particle from (1, –2, 1) to (3, 1, 4). *5*

(ii) Show that, $\nabla^2 f(r) = \left(\frac{2}{r}\right)f'(r) + f''(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. *5*

(b) Use divergence theorem to evaluate,

$$\iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dy\, dx),$$

where S is the sphere, $x^2 + y^2 + z^2 = 1$. 10

(c) If $\vec{A} = 2y\,\vec{i} - z\,\vec{j} - x^2\,\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, $z = 6$, evaluate the surface integral,

$$\iint_S \vec{A} \cdot \hat{n}\, \vec{dS}.$$

10

(d) Use Green's theorem in a plane to evaluate the integral, $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, where C is the boundary of the surface in the xy-plane enclosed by, $y = 0$ and the semi-circle, $y = \sqrt{1 - x^2}$. 10

IFoS 2010

Sl. No. 8951 **B–JGT–K–NBB**

MATHEMATICS

Paper—II

Time Allowed : Three Hours	Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks.

The number of marks carried by each part of a question is indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Symbols and notations have their usual meanings, unless indicated otherwise.

Section—A

1. Answer any *four* parts from the following :

(a) Let

$$G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \, a \in \mathbb{R}, \, a \neq 0 \right\}$$

Show that G is a group under matrix multiplication. 10

(b) Let F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$. 10

(c) If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f(x+y) = f(x) f(y)$$

for all x, y in $\mathbb{R}$ and $f(x) \neq 0$ for any x in $\mathbb{R}$, show that $f'(x) = f(x)$ for all x in $\mathbb{R}$ given that $f'(0) = f(0)$ and the function is differentiable for all x in $\mathbb{R}$. 10

(d) Determine the analytic function $f(z) = u + iv$ if $v = e^x (x \sin y + y \cos y)$. 10

(e) A captain of a cricket team has to allot four middle-order batting positions to four batsmen. The average number of runs scored by each batsman at these positions are as follows. Assign each batsman his batting position for *maximum* performance : 10

Batsman \ *Batting position*	*IV*	*V*	*VI*	*VII*
A	40	25	20	35
B	36	30	24	40
C	38	30	18	40
D	40	23	15	33

2. (a) A rectangular box open at the top is to have a surface area of 12 square units. Find the dimensions of the box so that the volume is maximum. 13

(b) Prove or disprove that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic groups where $\mathbb{R}^+$ denotes the set of all positive real numbers. 13

(c) Using the method of contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}$$ 14

3. (a) Show that zero and unity are only idempotents of Z_n if $n = p^r$, where p is a prime. 13

(b) Evaluate

$$\iint_R (x - y + 1)\, dx\, dy$$

where R is the region inside the unit square in which $x + y \geq \frac{1}{2}$. 13

(c) Solve the following linear programming problem by the simplex method : 14

Maximize $Z = 3x_1 + 4x_2 + x_3$

subject to

$$x_1 + 2x_2 + 7x_3 \leq 8$$
$$x_1 + x_2 - 2x_3 \leq 6$$
$$x_1,\ x_2, x_3 \geq 0$$

4\. *(a)* Let R be a Euclidean domain with Euclidean valuation d. Let n be an integer such that $d(1) + n \geq 0$. Show that the function $d_n : R - \{0\} \to S$, where S is the set of all negative integers defined by $d_n(a) = d(a) + n$ for all $a \in R - \{0\}$ is a Euclidean valuation. 13

(b) Obtain Laurent's series expansion of the function

$$f(z) = \frac{1}{(z+1)(z+3)}$$

in the region $0 < |z+1| < 2$. 13

(c) *ABC* Electricals manufactures and sells two models of lamps, L_1 and L_2, the profit per unit being Rs 50 and Rs 30, respectively. The process involves two workers W_1 and W_2, who are available for 40 hours and 30 hours per week, respectively. W_1 assembles each unit of L_1 in 30 minutes and that of L_2 in 40 minutes. W_2 paints each unit of L_1 in 30 minutes and that of L_2 in 15 minutes. Assuming that all lamps made can be sold, determine the weekly production figures that maximize the profit. 14

Section—B

5. Answer any *four* parts from the following :

(a) Find the general solution of

$$x(y^2+z)p+y(x^2+z)q=z(x^2-y^2)$$ 10

(b) Solve $x\log_{10}x=1\cdot2$ by regula falsi method. 10

(c) Convert the following : 10

(i) $(736\cdot4)_8$ to decimal number

(ii) $(41\cdot6875)_{10}$ to binary number

(iii) $(101101)_2$ to decimal number

(iv) $(AF63)_{16}$ to decimal number

(v) $(101111011111)_2$ to hexadecimal number

(d) Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same. 10

(e) A two-dimensional flow field is given by $\psi = xy$. Show that—

(i) the flow is irrotational;

(ii) ψ and ϕ satisfy Laplace equation.

Symbols ψ and ϕ convey the usual meaning. 10

6. *(a)* Using Lagrange interpolation, obtain an approximate value of $\sin(0 \cdot 15)$ and a bound on the truncation error for the given data : 12

$$\sin(0 \cdot 1) = 0 \cdot 09983, \ \sin(0 \cdot 2) = 0 \cdot 19867$$

(b) Draw a flow chart for finding the roots of the quadratic equation $ax^2 + bx + c = 0$. 12

(c) Solve

$$\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$$

given the conditions

(i) $u(0, t) = u(\pi, t) = 0, \quad t > 0$

(ii) $u(x, 0) = \sin 2x, \quad 0 < x < \pi$ 16

7. *(a)* Find the general solution of

$$(D - D' - 1)(D - D' - 2)z = e^{2x-y} + \sin(3x + 2y)$$

13

(b) Show that $\phi = (x - t)(y - t)$ represents the velocity potential of an incompressible two-dimensional fluid. Further show that the streamlines at time t are the curves

$$(x - t)^2 - (y - t)^2 = \text{constant}$$ 13

(c) Find the interpolating polynomial for (0, 2), (1, 3), (2, 12) and (5, 147). 14

8. *(a)* A mass m_1, hanging at the end of a string, draws a mass m_2 along the surface of a smooth table. If the mass on the table be doubled, the tension of the string is increased by one-half. Show that $m_1 : m_2 = 2 : 1$. 13

(b) Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1$$

for $x = 0 \cdot 1$ by Euler's method. 13

(c) Show that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving under no external forces satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \nabla)\vec{q} + \nu \nabla^2 \vec{\Omega}$$

where ν is the kinematic viscosity. 14

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