MATHEMATICS

PAPER - I

SECTION A

1. Answer any four of the following questions:

(a) Let W be the subspace of $\mathbf{R}^4$ generated by the vectors (1, -2, 5, -3), (2, 3, 1, -4) and (3, 8, -3, -5).

Find a basis and dimension of W.

(10)

(b) Find the symmetric matrix which corresponds to the following quadratic polynomial:

(10)

$$q(x, y, z) = x^2 - 2yz + xz$$

(c) Show that

$$\int_0^\infty \frac{\tan^{-1}\alpha x \tan^{-1}\beta x}{x^2}\,dx = \frac{\pi}{2}\log\left[\frac{(\alpha+\beta)^{(\alpha+\beta)}}{\alpha^\alpha+\beta^\beta}\right], \alpha, \beta > 0$$

(10)

(d) Show that

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

where

$f(x, y) = 0$, if $xy = 0$, $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, if $xy \neq 0$

(e) Prove that the locus of a line which meets the two lines $y = \pm mx$, $z = \pm c$ and the circle $x^2 + y^2 = a^2$, $z = 0$ is

$$c^2m^2(cy - mzx)^2 + c^2(yz - cmx)^2 = a^2m^2(z^2 - c^2)^2.$$

(10)

2. (a) Let

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Find all eigenvalues of A and the corresponding eigenvectors.

(10)

(b) Let

$$H = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix}$$

be a Herrhitian matrix. Find a non-singular matrix P such that $P^t H\overline{P}$ is diagonal.

(10)

(c) Let L : $\mathbf{R}^3 \rightarrow \mathbf{R}^2$ be a linear transformation for which we know that

L (1, 0, 0) = (2, -1), L(0, 1, 0) = (3, 1) and

L(0, 0, 1) = (-1, 2).

Find L (-3, 4, 2).

(10)

(d) Let L : $\mathbf{R}^3 \to \mathbf{R}^2$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \end{bmatrix}.$$

Let S = [v_1, v_2, v_3] and T = [w_1, w_2] be bases for $\mathbf{R}^3$ and $\mathbf{R}^2$ respectively, where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find the matrix of L with respect to S and T.

(10)

3. (a) Find the volume under the spherical surface $x^2 + y^2 + z^2 = a^2$ and over the lemniscate $r^2 = a^2 \cos 2\theta$.

(10)

(b) Find the centre of gravity of the volume common to a cone of vertical angle 2α and a sphere of radius a, the vertex of the cone being the centre of the sphere.

(10)

(c) Using Lagrange's method of undetermined multipliers, find the stationary values of $x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$.

Interpret geometrically.

(10)

(d) Find the extreme values of

$f(x, y) = 2(x - y)^2 - x^4 - y^4$.

(10)

4. (a) Two straight lines

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}; \quad \frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$$

are cut by a third line whose direction cosines are λ, μ and ν. Show that the length d intercepted on the third line is given by

$$d\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

Deduce the length of the shortest distance between the first two lines.

(16)

(b) Find the condition that the plane $lx + my + nz = 0$ be a tangent plane to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

(12)

(c) Prove that the locus of the pole of the plane $lx + my + nz = p$ with respect to system of conicoids $\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1$, where k is a parameter, is a straight line perpendicular to the given plane.

(12)

SECTION B

5. Answer any four of the following questions:

(a) From $x^2 + y^2 + 2ax + 2by + c = 0$, derive differential equation not containing a, b or c.

(10)

(b) Discuss the solution of the differential equation

$$y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=a^2,$$

(10)

(c) A particle moving in a straight line is acted on by a force which works at a constant rate and changes its velocity from u to v in passing over a distance x. Prove that the time taken is

$$\frac{3(u+v)x}{2\left(u^2+uv+v^2\right)}$$

(10)

(d) Find the work done in stretching an elastic string from length b to length c, the unstretched length of the string being a.

(10)

(e) If $\vec{f} = 3xy\hat{i} - y^2\hat{j}$, determine the value of $\int_C \vec{f}\cdot d\vec{r}$, where C is the curve $y = 2x^2$ in the xy-plane from (0, 0) to (1, 2).

(10)

6. (a) Solve

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}-y=e^x.$$

(10)

(b) Solve

$$\frac{d^4y}{dx^4}-y=x\sin x$$

(10)

(c) Solve

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x.$$

(10)

(d) Reduce

$$xy\left(\frac{dy}{dx}\right)^2-\left(x^2+y^2+1\right)\frac{dy}{dx}+xy=0$$

to Clairaut's form and find its singular solution.

(10)

7. (a) A particle is projected from an apse at distance 'c' with velocity $\sqrt{2\mu/3}\;c^3$. If the force directed to the centre be $\mu\left(r^5-c^4r\right)$, determine the equation of the orbit (μ being a constant).

(14)

(b) A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall. If the beam is in equilibrium in any position, find the equation to the curve.

(13)

(c) A solid right circular cone of vertical angle 2α is just immersed in water so that one generator is on the surface of the liquid. Determine the ratio between the resultant thrust on the curved surface of the cone and the weight of water displaced by the cone. Also find the angle between the resultant thrust and the axis of the cone.

(13)

8. (a) If $u\vec{f} = \vec{\nabla} v$, where u, v are scalar fields and $\vec{f}$ is a vector field, find the value of $\vec{f}$. curl $\vec{f}$.

(10)

(b) If O be the origin, A, B, two fixed points and P(x, y, z) a variable point, show that

$$\text{curl}\left(\vec{PA} \times \vec{PB}\right) = 2\,\vec{AB}$$

(10)

(c) Using Stokes theorem, determine the value of the integral

$$\int_{\Gamma} (y\,dx + z\,dy + x\,dz)$$

where Γ is the curve defined by

$$x^2 + y^2 + z^2 = a^2,\; x + z = a$$

(10)

(d) Prove that the cylindrical co-ordinate system is orthogonal.

(10)

MATHEMATICS

PAPER - II

SECTION A

1. Answer any four parts:

(a) If $f(z)$ is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2,\ z = x + iy$$

(10)

(b) Evaluate the double integral

$$\iint_R x^2 dxdy$$

where R is the region bounded by the line $y = x$ and the curve $y = x^2$

(10)

(c) Find the basic feasible solutions of the following system of equations in a linear programming problem :

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

(10)

(d) Show that the set of cube roots of unity is a finite Abelian group with respect to multiplication

(10)

(e) Show that the function f(x) defined by

$$f(x) = \frac{1}{x},\ x \in [1, \infty)$$

is uniformly continuous on $[1, \infty)$.

(10)

2. (a) Prove that the set of all real numbers of the form $\left(a + b\sqrt{b}\right)$, where a and b are rational numbers, is a field under usual addition and multiplication.

(14)

(b) Show that the transformation

$$\omega = \frac{5-4z}{4z-2}$$

maps the unit circle $|z| = 1$ into a circle of radius unity and centre at 1/2.

(13)

(c) Show that the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{\left(1+x^4\right)^2} + \ldots..$$

is not uniformly convergent on [0, 1].

(13)

3. (a) Examine the following function for extrema :

(13)

$$f(x_1, x_2) = x_1^3 - 6x_1x_2 + 3x_2^2 - 24x_1 + 4$$

(b) Use contour integration technique to find the value of

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$$

(13)

(c) Use Simplex method to solve the following linear programming problem

Maximize $Z\ 2x_1 - x_2 + 3x_3$

subject to constraints

$$\begin{aligned} 3x_1 + x_2 - 2x_3 &\le 6 \\ 2x_1 + 5x_2 + x_3 &\le 14 \\ x_1 + 4x_2 + 2x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

4. (a) Show that

(i) $h(x) = \sqrt{x + \sqrt{x}}, x \ge 0$ is continuous on $[0, \infty)$

(ii) $h(x) = e^{\sin x}$ is continuous on **R**

(13)

(b) Show that the set S = {1, 2, 3, 4} forms an Abelian group under the operation of multiplication modulo 5 as defined below:

(13)

x mod 5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(c) If $f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{where } x, y \ne 0 \\ 0, & \text{where } x = y = 0 \end{cases}$

show that at $(0,0)\ \dfrac{\partial^2 f}{\partial x \partial y} \ne \dfrac{\partial^2 f}{\partial y \partial y}$

(14)

SECTION B

5. Answer any four parts

(a) Perform four iterations of the bisection method to obtain a positive root of the equation

$$f(x) = x^3 - 5x + 1$$

(10)

(b) Apply Charpit's method to solve the equation

$$2z + p^2 + qy + 2y^2 = 0$$

(10)

(c) (i) Convert 239870 decimal to —

(1) Octal number;

(2) Hexadecimal number.

(ii) Evaluate the following expressions bitwise

(1) 78 OR 87

(2) 78 XOR 87

(3) 78 AND 87

(10)

(d) Determine the restrictions on f_1, f_2 and f_3 if

$$\frac{x^2}{a^2} f_1(t) + \frac{y^2}{b^2} f_2(t) + \frac{z^2}{c^2} f_3(t) = 1$$

is a possible boundary surface of a liquid.

(10)

(e) Find the moment of inertia of a uniform triangular lamina about one side.

(10)

6. (a) Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

given that

(*i*) $u = 0$ when $x = 0$ for all t

(*ii*) $u = 0$ when $x = l$ for all t

(*iii*) $u = \frac{bx}{a}$, $0 < x < a$, $t = 0$

$= \frac{b(l-x)}{l-a}$, $a < x < l$, $t = 0$

(*iv*) $\frac{\partial u}{\partial t} = 0$ at $t = 0$, x in $(0, l)$

(14)

(b) Write a BASIC program to evaluate a definite integral by Simpson's one-third rule. Adapt it to evaluate

$$\int_0^1 \left(x^3 + \sin x\right) dx$$

by taking 10 subintervals and indicating which lines are to be modified for a specific problem.

(13)

(c) Apply Gauss-Seidel iteration method for three iterations to solve the equation

$$\begin{bmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 27 \\ -9 \end{bmatrix}$$

(13)

7. (a) A two-dimensional flow field is given by $\psi = xy$.

(i) Show that the flow is irrigational.

(ii) Find the velocity potential.

(iii) Verify that ψ and its complex conjugate ϕ satisfy the Laplace equation.

(iv) Find the streamlines and potential lines.

(14)

(b) Solve:

$$\frac{\partial^2 z}{\partial x^2}-3\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=e^{2x+3y}+\sin(x-2y)$$

(c) Apply Runge-Kutta method of fourth order to find an approximate value of y when x = 0.1 and 0.2, given that

$$\frac{dy}{dx}=x+y^2,\ y=1 \text{ when } x=0$$

(13)

8. (a) A solid circular cylinder of radius a rotating about its axis is placed gently with its axis horizontal on a rough plane, whose inclination to the horizon is cc Initially the friction acts up the plane and the coefficient of friction is μ. Show that the cylinder will move upwards, if $\mu > \tan\alpha$. Also show that the time that lapses before the rolling commences is

$$\frac{a\Omega}{g\,(3\mu\cos\alpha-\sin\alpha)}$$

where Ω is the initial angular velocity of the cylinder.

(14)

(b) Write a program in BASIC to find a root of an equation by Newton-Raphson method. Adapt it to solve

$$x^3-4x^2+x+6=0$$

using initial approximation as $x_0 = 5$. Indicate which lines are to be changed for a different equation.

(13)

(c) Apply Newton's forward and backward difference formulae to evaluate $f(1.2)$ and $f(3.9)$ respectively from the data :

x	:	0	1	2	3	4
$f(x)$	:	1	1.5	2.2.	3.1	4.3

(13)