MATHEMATICS

PAPER - I

SECTION A

1. Answer any four of the following:

(a) Let V $P_3(R)$ be the vector space of polynomial functions on reals of degree at most 3. Let D: $V \to V$ be the differention operator defined by

$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$

(i) Show that D is a linear transformation

(ii) Find kernel and image of D.

(iii) What are dimensions of V, ker D and image D ?

(iv) Give relation among them of (iii). (10)

(b) Find the eigen values and the corresponding eigen vectors of $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ (10)

(c) Let f, g: $[a, b] \to IR$ be functions such that $f'(x)$ and $g'(x)$ exist for all $x \in [a, b]$ and $g'(x) \neq 0$ for all x in (a, b). Prove that for some $c \in (a, b)$

$$\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$$

(10)

(d) Let f: $IR^2 \to IR$ be defined by

$$f(x, y) = \frac{xy^2}{x^2+y^2} \text{ for } (x, y) \neq (0,0)$$

$= 0$ for $(x, y) = (0, 0)$

Show that the partial derivatives $D_1 f(0, 0)$ and $D_2 f(0, 0)$ vanish but f is not differentiable at (0, 0).

(10)

(e) A line makes angles α, β, γ, δ with the four diagonals of a cube. Show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

(10)

2. (a) Show that the vectors (1, 2, 1), (1, 0, —1) and (0, —3, 2) form a basis for $R^{(3)}$

(10)

(b) Determine non-singular matrices P and Q such that the matrix PAQ is in canonical form, where

$$A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$$

Hence find the rank of A.

(10)

(c) Find the minimum polynomial of the matrix.

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

and use it to determine whether A is similar to a diagonal matrix.

(10)

(d) Show that the quadratic form

$2x_2—4xy + 3xy + 6y^2 + 6yz + 8z^2$

in three variables is positive definite.

(10)

3. (a) Let $f(x) = e^{-1/x^2}$ $(x \neq 0)$

$= 0$ for $x = 0$

Show that $f'(0) = 0$ and $f''(0) = 0$.

Write $f^{(k)}(x)$ as $P\left(\frac{1}{x}\right)f(x)$ for $x \neq 0$, where P is a polynomial and $f^{(k)}$ denotes the k^{th} derivation of f.

(10)

(b) Using Lagrange multipliers, show that a rectangular box with lid of volume 1000 cubic units and of least surface area is a cube of side 10 units.

(10)

(c) Show that the area of the surface of the solid obtained by revolving the arc of the curve $y = c \cosh\left(\frac{x}{c}\right)$ joining (0, c) and (x, y) about the x-axis is

(10)

(d) Define $\Gamma : (0, \infty) \to \mathrm{IR}$ by

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Show that this integral converges for all $x > 0$ and that $\Gamma(x + 1) = x\,\Gamma(x)$.

(10)

4. (a) Show that the equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represents a cone that touches the co-ordinate planes and that the equation to its reciprocal cone is

$fyz + gzx + hxy = 0$

(10)

(b) Show that any two generators belonging to the different system of generating lins of a hyperboloid of one sheet intersect.

(10)

(c) Show that the locus of a point from which three mutually perpendicular tangent lines can be drawn to the paraboloid

$ax^2 + by^2 + 2z = 0$ is

$ab(x^2 + y^2) + 2(a + b)z = 1.$

(10)

(d) Show that the enveloping cylinder of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

whose generators are parallel to the line

$$\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$$

meet the plane $z = 0$ in circles.

(10)

SECTION B

5. Answer any four of the following:

(a) Find the orthogonal trajectories of the family of co-axial circles

$x^2 + y^2 + 2gx + c = 0$

where g is a parameter (10)

(b) Find three solutions of

$$\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0$$

which are linearly independent on every real interval. (10)

(c) A particle moves with an acceleration which is always towards, and equal to μ divided by the distance from, a fixed point 0. If it starts from rest at a distance a from 0, show that it will arrive at 0 in time

$$a\sqrt{\frac{\pi}{2\mu}}.$$

(10)

(d) Show that the depth of the centre of pressure of the area included between the arc and the asymptote of the curve

$$(r-a)\cos\theta=b \text{ is } \frac{a}{4}\cdot\frac{3\pi a+16b}{3\pi b+4a},$$

the asymptote being in the surface and the plane of the curve being vertical. (10)

(e) Find expressions for curvature and torsion at a point on the curve $x = a\cos\theta$ $y = a\sin\theta$, $z = a\theta\cot\beta$ (10)

6. (a) Solve and examine for singular solution:

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2.$$

(10)

(b) Solve

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right).$$

(10)

(c) Given y = x is one solutions of

$$(x^3+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$$

find another linearly independent solution by reducing order and write the general solution.

(10)

(d) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2}+a^2y=\sec ax,\ a \text{ is real.}$$

(10)

7. (a) A shell fired with velocity V at an elevation θ hits an airship at a height h from the ground, which is moving, horizontally away from the gun with velocity v. Show that if

$$(2\,V\cos\theta-v)\,(v^2\sin^2\theta-2gh)^{1/2}=v\,V\sin\theta$$

the shell might have also hit the ship if the latter had remained stationary in the position it occupied when the gun was actually fired. (10)

(b) Assuming the eccentricity e of a planet's orbit is a small fraction, show that the ratio of the time taken by the planet to travel over the halves of its orbit separated by the minor axis is nearly $1+\frac{4e}{\pi}$ (10)

(c) A uniform rod AB of length 2a is hinged at A, a string attached to the middle point G of the rod passes over a smooth pulley at C at a height a, vertically above A, and supports a weight P having freely, find the positions of equilibrium and determine their nature as to stability or unstability (10)

(d) A solid of cork bounded by the surface generated by the revolution of a quadrant of an ellipse about the major axis sinks in mercury up to the focus. If the centre of gravity of the cork coincides with the metacentre, prove that $2e^4+4e^3+2e^2-e-2=0$ (10)

8. (a) If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, prove that

$$\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r).$$

Find f(r) such that $\Delta^2 f(r)=0$ (10)

(b) If $\vec{F}$ is solenoidal, prove that Curl Curl Curl Curl $\vec{F}=\nabla^4\vec{F}$ (10)

(c) Verify Stoke's Theorem when

$$\bar{F}=(2\,xy-x^2)\,\bar{i}-(x^2-y^2)\,\bar{j}$$

and C is the boundary of the region enclosed by the parabolas $y^2=x$ and x2 = y. (10)

(d) Express $\nabla\times\vec{F}$ and $\nabla^2\Phi$ in cylindrical co-ordinates, (10)

MATHEMATICS

PAPER - II

SECTION A

1. Answer any four parts: (10 x 4 = 40)

(a) Show that if every element of a group (G, *) be its own inverse, then it is an Abelian group. Give an example to show that the converse is not true.

(b) Evaluate

$$I = \iint (a^2 - x^2 - y^2)^{1/2} dx\, dy$$

over the positive quadrant of the circle

$x^2 + y^2 = a^2$

(c) If $w = f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is analytic in a domain, show that

$$\frac{\partial w}{\partial \bar{z}} = 0$$

Hence or otherwise, show that $\sin(x + i3y)$ cannot be analytic.

(d) Investigate the continuity of the function $f(x) = [x]/x$ for $x \neq 0$ and $f(0) = -1$.

(e) (i) Explain the following terms of an LPP:

1. Solution
2. Basic solution
3. Basic feasible solution
4. Degenerate basic solution

(ii) Give the dual of the following LFP:

Maximize $Z = 2x_1 + 3x_2 + x_3$

Subject to $4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 3x_3 = 4$

$x, x_2, x_3 \geq 0$

2. (a) Let $G = \{a \in R: -1 < a < 1\}$. Define a binary operation * on G by

$$a * b = \frac{a+b}{1+ab} \text{ for all } a, b \in G$$

Show that (G, *) is a group

(13)

(b) Let $f(x) = [x]$, $x \in [0, 3]$, where [x] denotes the greatest integer not greater than x. Prove that f is Riemann integrable on [0, 3] and evaluate

$$\int_0^3 f(x)\,dx$$

(13)

(c) (i) Let (a, b) be any open interval, f a function defined and differentiable on (a, b) such that its derivative is bounded on (a, b). Show that f is uniformly continuous on (a, b).

(ii) If f is a continuous function on [a, b] and if

$$\int_a^b f^2(x)\,dx = 0$$

then show that fix) = 0 for all x in [a, b]. Is this true if f is not continuous ?

(14)

3. (a) Let R be the set of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ a, b \in F$$

where F is a field. With usual addition and multiplication as binary operations, show that R is a commutative ring with unity. Is it a field if $F = Z_2, Z_5$?

(14)

(b) Discuss the transformation

$$w = z + \frac{1}{z}$$

and hence, show that—

(i) a circle in z-plane is mapped on an ellipse in the w-plane;

(ii) a line in the z-plane is mapped into a hyperbola in the w-plane.

(13)

(c) Find the Laurent series expansion of the function

$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$$

valid in the region $2 < |z| < 3$.

(13)

4. (a) Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

(13)

(b) Find the maximum value of $Z = 2x + 3y$ subject to the constraints

$$\begin{aligned} x - y &\geq 0 \\ x + y &\leq 30 \\ y &\geq 3 \\ 0 \leq y &\leq 12 \text{ and} \\ 0 \leq x &\leq 20 \end{aligned}$$

by graphical method.

(13)

(c) Apply simplex method to solve the following linear programming problem

(14)

Maximize $Z = 4x_1 + 3x_2$ subject to the constraints

$$\begin{aligned} &3x_1 + x_2 \le 15 \\ &3x_1 + 4x_2 \le 24 \\ &x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

SECTION B

5. Answer any four parts:

(10 x 4 = 40)

(a) Find the general solution of the partial differential equation

$$(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$$

(10)

(b) Find the cube root of 10 using Newton-Raphson method, correct to 4 decimal places.

(10)

(c) Apply modified Euler's method to determine y (0.1), given that

$$\frac{dy}{dx} = x^2 + y$$

when y (0) = 1.

(10)

(d) (i) Convert ABCD hex and 76543 octal to decimal.

(ii) Convert 39870 decimal to octal and hexadecimal.

(10)

(e) Show that the surface

$$\frac{x^2}{a^2k^2t^4} + kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$$

is a possible form of boundary surface of a liquid at time t.

(10)

6. (a) Form the partial differential equation by eliminating the arbitrary function from

$$f(x^2 + y^2,\ z - xy) = 0,\ z = z\ (z,\ y)$$

(13)

(b) Solve

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial t^2}$$

given that

(i) u = 0, when t = 0 for all t

(ii) u = 0 when x = l for all t

(iii) $\left.\begin{aligned} u &= x \quad in\left(0, \frac{l}{2}\right) \\ &= l - x \quad in\left(\frac{l}{2}, l\right) \end{aligned}\right\} at\ t = 0$

(14)

(c) A rod of length 2 a is suspended by a string of length l attached to one end; if the string and rod revolve about the vertical with uniform angular velocity and their inclinations to the vertical be θ and ϕ, respectively, show that

$$\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$$

(13)

7. (a) The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time (in min) :	0	2	4	6	8	10	12
Velocity (in km/hr) :	0	22	30	27	18	7	0

Apply Simpson's one-third rule to find the distance covered by the car.

(13)

(b) Consider the velocity field given by

$$\vec{q} = i(1 + At) + jx$$

Find the equation of the streamline at $t = t_0$ passing through the point (x_0, y_0). Also, obtain the equation of the path line of a fluid element which comes to (x_0, y_0) at $t = t_0$. Show that, of $A = 0$, the streamline and path line coincide.

(13)

(c) Write a program in BASIC to integrate

$$\int_0^{10}\left(1 - e^{-\frac{x}{2}}\right)dx$$

by trapezoidal rule for 20 equal sub-divisions of the interval (0, 10). Indicate which lines are to be changed for a different integral.

(14)

8. (a) Draw a flowchart and write a program in BASIC for an algorithm to determine the greatest common divisor of two given positive integers.

(13)

(b) Apply Runge-Kutta method of order 4 to find an approximate value of y when x = 0.2 given that

$$\frac{dy}{dx} = x + y,\ y = 1 \text{ when } x = 0.$$

(14)

(c) A uniform sphere rolls down an inclined plane rough enough to prevent any sliding; discuss the motion. Hence, show that for pure rolling μ (coefficient of friction) is greater than $\frac{2}{7}$ tan α for a solid sphere, where α the inclination of the plane.

(13)