C. S. (Main) Exam : 2011

Serial No. | **C–DTN–L–NUA**

MATHEMATICS

Paper—I

Time Allowed : Three Hours | *Maximum Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

*Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any **three** of the remaining questions selecting at least **one** question from each Section.*

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

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SECTION—A

1\. (a) Let A be a non-singular, $n \times n$ square matrix. Show that $A \cdot (\text{adj } A) = |A| \cdot I_n$. Hence show that $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$. 10

(b) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$.

Solve the system of equations given by

$$AX = B$$

Using the above, also solve the system of equations $A^T X = B$ where A^T denotes the transpose of matrix A. 10

(c) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^3 + y^3}$ if it exists. 10

(d) Let f be a function defined on $\mathbb{R}$ such that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x in $\mathbb{R}$. How large can f(2) possibly be ? 10

(e) Find the equations of the straight line through the point (3, 1, 2) to intersect the straight line

$$x + 4 = y + 1 = 2(z - 2)$$

and parallel to the plane $4x + y + 5z = 0$. 10

(f) Show that the equation of the sphere which touches the sphere

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$$

at the point (1, 2, –2) and passes through the point (–1, 0, 0) is

$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0.$$ 10

2. (a) (i) Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigen values of a $n \times n$ square matrix A with corresponding eigen vectors $X_1, X_2, ..., X_n$. If B is a matrix similar to A show that the eigen values of B are same as that of A. Also find the relation between the eigen vectors of B and eigen vectors of A. 10

(ii) Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}.$$

Using this, show that A is non-singular and find A^{-1}. 10

(b) (i) Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors {(1, 1, –1), (1, 0, 1)} and {(1, 2, –3), (5, 2, 1)} are identical. Also find the dimension of this subspace. 10

(ii) Find the nullity and a basis of the null space of the linear transformation $A : R^{(4)} \to R^{(4)}$ given by the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$ 10

(c) (i) Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T : \mathbb{R}^{(3)} \to \mathbb{R}^{(3)}$ be a linear transformation defined by

$$T(x, y, z) = (x + 2y + 3z,\ x + 2y + 5z,\ 2x + 4y + 6z).$$

Show that the images of above vectors under T are linearly dependent. Give the reason for the same. 10

(ii) Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of the matrix B^3 where $B = C^{-1} AC$. 10

3. (a) Evaluate :

(i) $\lim_{x \to 2} f(x)$, where $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ \pi \quad , & x = 2 \end{cases}$

(ii) $\int_0^1 \ell n\, x\, dx$. (8, 12)

(b) Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$. 20

(c) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$. 20

4. (a) Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere. 20

(b) Show that the cone $yz + zx + xy = 0$ cuts the sphere $x^2 + y^2 + z^2 = a^2$ in two equal circles, and find their area. 20

(c) Show that the generators through any one of the ends of an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2 + b^2 = 6c^2$. Find also the condition for the generators to be perpendicular to each other. 20

SECTION—B

5. (a) Obtain the solution of the ordinary differential equation $\frac{dy}{dx}=(4x+y+1)^2$,

if $y(0) = 1$. 10

(b) Determine the orthogonal trajectory of a family of curves represented by the polar equation

$$r = a(1 - \cos\theta),$$

(r, θ) being the plane polar coordinates of any point. 10

(c) The velocity of a train increases from 0 to v at a constant acceleration f_1, then remains constant for an interval and again decreases to 0 at a constant retardation f_2. If the total distance described is x, find the total time taken. 10

(d) A projectile aimed at a mark which is in the horizontal plane through the point of projection, falls x meter short of it when the angle of projection is α and goes y meter beyond when the angle of projection is β. If the velocity of projection is assumed same in all cases, find the correct angle of projection.

10

(e) For two vectors $\vec{a}$ and $\vec{b}$ given respectively by

$$\vec{a} = 5t^2\hat{i} + t\,\hat{j} - t^3\hat{k}$$

and $\vec{b} = \sin t\,\hat{i} - \cos t\,\hat{j}$

determine :

(i) $\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$

and (ii) $\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)$. 10

(f) If u and v are two scalar fields and $\vec{f}$ is a vector field, such that

$$u\,\vec{f} = \text{grad } v,$$

find the value of

$\vec{f}\cdot\text{curl }\vec{f}$ 10

6. (a) Obtain Clairaut's form of the differential equation

$$\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right)=a^2\frac{dy}{dx}.$$

Also find its general solution. 15

(b) Obtain the general solution of the second order ordinary differential equation

$$y'' - 2y' + 2y = x + e^x \cos x,$$

where dashes denote derivatives w.r. to x. 15

(c) Using the method of variation of parameters, solve the second order differential equation

$$\frac{d^2y}{dx^2}+4y=\tan 2x.$$

15

(d) Use Laplace transform method to solve the following initial value problem :

$$\frac{d^2x}{dt^2}-2\frac{dx}{dt}+x=e^t,\ x(0)=2 \text{ and } \left.\frac{dx}{dt}\right|_{t=0}=-1.$$

15

7. (a) A mass of 560 kg. moving with a velocity of 240 m/sec strikes a fixed target and is brought to rest in $\frac{1}{100}$ sec. Find the impulse of the blow on the target and assuming the resistance to be uniform throughout the time taken by the body in coming to rest, find the distance through which it penetrates. 20

(b) A ladder of weight W rests with one end against a smooth vertical wall and the other end rests on a smooth floor. If the inclination of the ladder to the horizon is 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from slipping down. 20

(c) (i) After a ball has been falling under gravity for 5 seconds it passes through a pane of glass and loses half its velocity. If it now reaches the ground in 1 second, find the height of glass above the ground. 10

(ii) A particle of mass m moves on straight line under an attractive force mn^2x towards a point O on the line, where x is the distance from O. If $x = a$ and $\frac{dx}{dt} = u$ when $t = 0$, find $x(t)$ for any time $t > 0$. 10

8. (a) Examine whether the vectors ∇u, ∇v and ∇w are coplanar, where u, v and w are the scalar functions defined by :

$$u = x + y + z,$$
$$v = x^2 + y^2 + z^2$$
and $$w = yz + zx + xy.$$ 15

(b) If $\vec{u} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$, calculate the double integral

$$\iint (\nabla \times \vec{u}) \cdot d\vec{s}$$

over the hemisphere given by

$$x^2 + y^2 + z^2 = a^2, \qquad z \geq 0.$$ 15

(c) If $\vec{r}$ be the position vector of a point, find the value(s) of n for which the vector

$$r^n \ \vec{r}$$

is (i) irrotational, (ii) solenoidal. 15

(d) Verify Gauss' Divergence Theorem for the vector

$$\vec{v} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

taken over the cube

$$0 \leq x, y, z \leq 1.$$ 15

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SECTION—A

1. (a) Show that the set

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

of six transformations on the set of Complex numbers defined by

$$f_1(z) = z, \; f_2(z) = 1 - z$$

$$f_3(z) = \frac{z}{(z-1)}, \; f_4(z) = \frac{1}{z}$$

$$f_5(z) = \frac{1}{(1-z)} \text{ and } f_6(z) = \frac{(z-1)}{z}$$

is a non-abelian group of order 6 w.r.t. composition of mappings. 12

(b) Let $S = (0, 1]$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \leq 1$ (in $\mathbb{R}$). Is f uniformly continuous on S ? Justify your answer. 12

(c) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \frac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, find f(z) subject to the condition, $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$. 12

(d) Solve by Simplex method, the following LP Problem :

Maximize, $Z = 5x_1 + 3x_2$

Constraints, $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$ 12

(e) (i) Prove that a group of Prime order is abelian. 6

(ii) How many generators are there of the cyclic group $(G, \cdot)$ of order 8 ? 6

2. (a) Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic. 15

(b) Let $f_n(x) = nx(1-x)^n$, $x \in [0, 1]$

Examine the uniform convergence of $\{f_n(x)\}$ on $[0, 1]$. 15

(c) If the function $f(z)$ is analytic and one valued in $|z - a| < R$, prove that for $0 < r < R$,

$f'(a) = \dfrac{1}{\pi r}\displaystyle\int_0^{2\pi} P(\theta)e^{-i\theta}\,d\theta$, where $P(\theta)$ is the real part of $f(a + re^{i\theta})$. 15

(d) Find the shortest distance from the origin (0, 0) to the hyperbola

$$x^2 + 8xy + 7y^2 = 225$$ 15

3. (a) Let F be the set of all real valued continuous functions defined on the closed interval [0, 1]. Prove that (F, +, ·) is a Commutative Ring with unity with respect to addition and multiplication of functions defined pointwise as below :

$$\left.\begin{array}{ll} & (f + g)(x) = f(x) + g(x) \\ \text{and} & (f \cdot g)(x) = f(x) \cdot g(x) \end{array}\right\} x \in [0, 1]$$

where $f, g \in F$. 15

(b) Show that the series for which the sum of first n terms

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad 0 \le x \le 1$$

cannot be differentiated term-by-term at $x = 0$. What happens at $x \neq 0$? 15

(c) Evaluate by Contour integration,

$$\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}}$$ 15

(d) Find the Laurent series for the function

$$f(z) = \frac{1}{1-z^2}, \text{ with centre } z = 1.$$ 15

4. (a) Let a and b be elements of a group, with $a^2 = e$, $b^6 = e$ and $ab = b^4a$.

Find the order of ab, and express its inverse in each of the forms $a^m b^n$ and $b^m a^n$. 20

(b) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then its derivative

$$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}, \text{ for all } x. \quad 20$$

(c) Write down the dual of the following LP problem and hence solve it by graphical method :

Minimize, $Z = 6x_1 + 4x_2$

Constraints, $2x_1 + x_2 \geq 1$

$3x_1 + 4x_2 \geq 1{\cdot}5$

$x_1, x_2 \geq 0$ 20

SECTION—B

5. (a) Solve the PDE

$$(D^2 - D'^2 + D + 3D' - 2)\, z = e^{(x-y)} - x^2 y$$

12

(b) Solve the PDE

$$(x + 2z)\frac{\partial z}{\partial x} + (4zx - y)\frac{\partial z}{\partial y} = 2x^2 + y$$

12

(c) Calculate $\int_2^{10} \frac{dx}{1+x}$ (upto 3 places of decimal) by dividing the range into 8 equal parts by Simpson's $\frac{1}{3}$rd Rule. 12

(d) (i) Compute $(3205)_{10}$ to the base 8.

(ii) Let A be an arbitrary but fixed Boolean algebra with operations $\wedge$, $\vee$ and $'$ and the zero and the unit element denoted by 0 and 1 respectively. Let x, y, z, ... be elements of A.

If $x, y \in A$ be such that $x \wedge y = 0$ and $x \vee y = 1$ then prove that $y = x'$. 12

(e) Let a be the radius of the base of a right circular cone of height h and mass M. Find the moment of inertia of that right circular cone about a line through the vertex perpendicular to the axis.

12

6. (a) Find the surface satisfying $\dfrac{\partial^2 z}{\partial x^2} = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$. 20

(b) Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le a,\ 0 \le y \le b$$

satisfying the boundary conditions

$u(0, y) = 0$, $u(x, 0) = 0$, $u(x, b) = 0$

$$\frac{\partial u}{\partial x}(a, y) = T \sin^3 \frac{\pi y}{a}.$$

20

(c) Obtain temperature distribution $y(x, t)$ in a uniform bar of unit length whose one end is kept at $10°C$ and the other end is insulated. Also it is given that $y(x, 0) = 1 - x, 0 < x < 1$. 20

7. (a) A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates :

x	·00	·25	·50	·75	1
y	1	·9896	·9589	·9089	·8415

Find the volume of the solid. 20

(b) Find the logic circuit that represents the following Boolean function. Find also an equivalent simpler circuit :

x	y	z	f(x, y, z)
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

20

(c) Draw a flow chart for Lagrange's interpolation formula. 20

(Contd.)

8. (a) The ends of a heavy rod of length 2a are rigidly attached to two light rings which can respectively slide on the thin smooth fixed horizontal and vertical wires O_x and O_y. The rod starts at an angle α to the horizon with an angular velocity $\sqrt{[3g(1-\sin\alpha)/2a]}$ and moves downwards. Show that it will strike the horizontal wire at the end of time

$$-2\sqrt{a/(3g)}\log\left[\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right].$$ 30

(b) An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength K but they are alternately of opposite signs. Find the Complex function that determines the velocity potential and the stream function. 30