MATHEMATICS

Time Allowed: 3 hours **Maximum Marks: 300**

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

Section-A

1. Attempt any five of the following:

(a) Let S be the vector space of all polynomials p(x), with real coefficients, of degree less than or equal to two considered over the real field IR, such that p(0) = 0 and p(1) = 0. Determine a basis for S and hence its dimension.

(12)

(b) Let T be the linear transformation from R^3 to R^4 defined by

$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$

For each $(x_1, x_2, x_3) \in R^3$.

Determine a basic for the Null space of T. What is the dimension of the Range space of T?

(12)

(c) Let f(x) $(x \in (x \in (-\pi, \pi)$? If it is continuous, then it is differentiable on $((-\pi, \pi)$?

(12)

(d) A figure bounded by one arch of a cycloid

$x = a (t – \sin t), y = a (1- \cos t), t \in [0, 2\pi]$, and the x-axis is revolved about the x-axis. Find the volume of the solid of revolution.

(12)

(e) Find the focus of the point which moves so that its distance from the plane x + y - z = 1 is twice its distance from the line x = - y = z.

(12)

(f) Find the equation of the sphere inscribed in the tetrahedron whose faces are x = 0, y = 0, z = 0 and 2x + 3y + 6z = 6.

(12)

2. (a) Let W be the set of all 3 x 3 symmetric matrices over R. Does it form a subs pace of the vector space of the 3 x 3 matrices over IR? In case it does, construct a basis for this space and determine its dimension.

(20)

(b) Consider the vector space

X: = {p(x) p(x) is a polynomial of degree less than or equal to 3 with real coefficients}, over the real field R, Define the map $D : X \to X$ by

$(Dp)(x) : p_1 + 2p_2x + 3p_3x^2$

where $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 +$

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the ordered basis $\{1, x, x2, x^3\}$ for X.

(20)

(c) Reduce the quadratic form q(x, y, z): $= x^2 + 2y^2 - 4xz + 4yz + 7z^2$ to canonical form. Is q positive definite?

(20)

3. (a) Find a rectangular parallelopiped of greatest volume for a given total surface area S, using Lagrange's method of multipliers.

(20)

(b) Prove that if $z = \psi(y + ax) + \psi(y - ax)$ then

$$a^2\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

for any twice differentiable φ and ψ; a is a constant.

(15)

(c) Show that $e^{-x}x^n$ is bounded on $[0, \infty)$ for all positive integral values of n. Using this result show that

$\int_0^\infty e^{-x}x^n dx$ exists.

(25)

4. (a) Show that the spheres $x^2 + y^2 + z^2 - x + z - 2 = 0$ and $3x^2 + 3y^2 - 8x - 10y + 8z + 14 = 0$ cut orthogonally. Find the centre and radius of their common circle.

(15)

(b) A line with direction ratios 2, 7, -5 is drawn to intersect the lines

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z-2}{4} \text{ and } \frac{x-11}{3} = \frac{y-5}{-1} = \frac{z}{1}$$

Find the coordinates of the points of intersection and the length intercepted on it.

(15)

(c) Show that the plane $2x - y + 2z = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines.

(15)

(d) Show that the feet of the normals from the point P(α, β, γ), β ≠ 0 on the paraboloid $x^2 + y^2 = 4z$ lie on the sphere

$$2\beta\left(x^2 + y^2 + z^2\right) - \left(\alpha^2 + \beta^2\right)y - 2\beta(2+\gamma)z = 0$$

(15)

7. (a) A particle is performing simple harmonic motion of period T about a centre 0. It passes through a point P (OP = p) with velocity v in the direction OP. Show that the time which elapses before it returns to P is

$$\frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi p}$$

(b) A particle attached to a fixed peg O by a string of length 1, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle 0 with the horizontal, the resultant acceleration is $g\sqrt{\left(1+3\sin^2\theta\right)}$

(15)

(c) A uniform beam of length 1 rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β (β > α), show that the inclination θ of the beam to the horizontal , in one of the equilibrium positions, is given by

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$$

& show that the beam is unstable in this position.

(15)

(d) A cone whose vertical angle is $\frac{\pi}{3}$, has its lowest generator horizontal and is filled with a liquid. Prove that the pressure on the curved surface is $\frac{W}{2}\sqrt{19}$ where W is the weight of the liquid.

(15)

8. (a) Find the curvature and torsion at any point of the curve x = a cos 2t, y = a sin 2t, z = 2a sin t.

(15)

(b) For any constant vector $\vec{a}$, show that the vector represented by curl $\left(\vec{a}\times\vec{r}\right)$ is always parallel to the vector $\vec{a}$, $\vec{r}$ being the position vector of a point (x, y, x) measured from the origin.

(15)

(c) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the value(s) of n in order than $r^n\,\vec{r}$ may be (i) solenoidal, (ii) irrational.

(15)

(d) Determine $\int_C (ydx + zdy = xdz)$ by using Stroke's theorem, where C is the curve defined by

$(x- a)^2 + (y - a)^2 + z^2 = 2a^2$, x + y = 2a

that starts from the point (2a, 0, 0) and goes at first below the z-plane.

(15)

MATHEMATICS

Time Allowed: 3 hours — ***Maximum Marks: 300***

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER I

SECTION - A

1. Answer any Five of the following:

(a) If in a group G, $a^5 = e$, e is the identity element of G $aba^{-1} = b^2$ for a, b $\in$ G, then find the order of b.

(12)

(b) Let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c d, $\in$ Z. Show that R is a ring under matrix addition and multiplication $A = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} a, b \in Z \right\}$

Then show that A is a left ideal of R but not a right ideal of R.

(12)

(c) Show that the function given by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

(12)

(d) Using Lagrange's mean value theorem, show that

$|\cos b - \cos a| \leq |b - a|$.

(12)

(c) Prove that the function f defined by

$$f(z) = \begin{cases} \dfrac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not differentiable at z = 0.

(f) Put the following in slack form and describe which of the variables are 0 at each of the vertices of the constraint set and hence determine the vertices algebraically:

Maximize

$u = 4x + 3y$

subject to

$x + y \le 4$

$-x + y \le 2$

$x, y \ge 0.$

(12)

2. (a) (i) Prove that there exists no simple group of order 48.

(15)

(ii) $1+\sqrt{-3}$ *and* $Z\left[\sqrt{-3}\right]$ is an irreducible element, but not prime. Justify your answer.

(15)

(b) Show that in the ring

$R = \left\{a + b\sqrt{-5} \mid a, b \in Z\right\},$

The element $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but αy and βy have no g.c.d. in R, where $\gamma = 7\left(1+2\sqrt{-5}\right)$.

(30)

3. (a) Given a positive real number a and any natural number n, prove that there exists one and only one positive real number ξ such that $\xi^n = a$.

(20)

(b) Find the volume of the solid in the octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$.

(20)

(c) Rearrange the series

$\sum_{n=1}^{\infty}(-1)^{n+1}.\frac{1}{n}$ to converge to 1.

(20)

4. (a) Evaluate (by using residue theorem)

$$\int_0^{2\pi}\frac{d\theta}{1+8\cos^2\theta}$$

(15)

(b) Show that the transformation $w = z^2$ conformal at point $z = 1 + i$ by finding the images of the lines $y = x$ and $x = 1$ which interest at $z = 1 + i$.

(15)

(c) Solve the following by Simplex method:

Maximize

$u = x + y$

subject to

$-x + y \le 1$

$x + 2y \le 4$

$x, y \ge 0.$

(30)

SECTION B

5. Answer any FIVE of the following:

(a) (i) Form a partial differential equation by eliminating the function f from :

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

(6)

(ii) Solve

$2zx - px^2 - 2qyx + pq = 0.$

(6)

(b) Transform the equation

$yz_x - xz_y = 0$

into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution.

(12)

(c) Use the method of false position to find a real root of $x^3 - 5x - 7 = 0$ lying between 2 and 3 and correct to 3 places of decimals.

(12)

(d) Convert :

(i) 46655 given to be in the decimal system into one in base 6,

(ii) $(11110.01)_2$ into a number in the decimal system.

(6+6)

(e) Consider a system with two degree of freedom for which

$V = q_1^2 + 3q_1q_2 + 4q_2^2.$

Find the equilibrium position and determine if the equilibrium is stable.

(12)

(f) Show that $\left(\frac{x^2}{a^2}\right)\cos^2 t + \left(\frac{y^2}{b^2}\right)\sec^2 t = 1$ is a possible form for the boundary surface of a liquid.

(12)

6. (a) Solve

$u_{xx} + u_{yy} = 0$ in D

where D $\{(x, y) : 0 < x < a, 0 < y < b\}$ is a rectangle in a plane with the boundary conditions:

$u(x, 0) = 0,\ u(x, b) = 0,\ 0 \le x \le a$

$u(0, y) = g(y),\ u_x(a, y) = h(y),\ 0 \le y \le b.$

(30)