C.S.E. (MAIN)
MATHEMATICS — 2006
PAPER-I

Time allowed : 3 hours *Maximum Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Questions 1 and 5 which are compulsory, and any **three** *of the remaining questions selecting at least* **one** *question from each Section.*

Assume suitable data if considered necessary and indicate the same clearly.

The number of marks carried by each question is indicated at the end of the question.

SECTION 'A'

Q. 1. Attempt any *five* of the following :

(a) Let V be the vector space of all 2 × 2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V. 12

(b) State Caylay-Hamilton theorem and using it, find the inverse of

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$ 12

(c) Find a and b so that f′ (2) exists, where

$$f(x) = \begin{cases} \dfrac{1}{|x|}, & \text{if } |x| > 2 \\ a + bx^2, & \text{if } |x| \le 2 \end{cases}$$ 12

(d) Express $\int_0^1 x^m (1 - x^n)^p \, dx$ in terms of Gamma function and hence evaluate the integral

$$\int_0^1 x^6 \sqrt{(1-x^2)} \, dx$$ 12

(e) A pair of tangents to the conic $ax^2 + by^2 = 1$ intercepts a constant distance 2 k on the y-axis. Prove that the locus of their point of intersection is the conic.

$$ax^2 (ax^2 + by^2 - 1) = bk^2 (ax^2 - 1)^2$$ 12

(f) Show that the length of the shortest distance between the line $z = x \tan \alpha$, $y = 0$ and any tangent to the ellipse $x^2 \sin^2 \alpha + y^2 = a^2$, $z = 0$ is constant. 12

Q. 2. (a) If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$T(x, y) = (2x - 3y, x + y)$$

compute the matrix of T relative to the basis

$$\mathscr{B} = \{(1, 2), (2, 3)\}$$ 15

(b) Using elementary row operations, find the rank of the matrix

$$\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$ 15

(c) Investigate for what values of λ and μ the equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have –

(i) no solution;
(ii) a unique solution;
(iii) infinitely many solutions. 15

(d) Find the quadratic form q (x, y) corresponding to the symmetric matrix

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix}$$

Is this quadratic form positive definite ? Justify your answer. 15

Q. 3. (a) Find the values of a and b such that

$$\lim_{x\to 0} \frac{a \sin^2 x \times b \log \cos x}{x^4} = \frac{1}{2}$$ 15

(b) If

$$z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$

show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2_{xy} \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$ 15

(c) Change the order of integration in

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$$

and hence evaluate it. 15

(d) Find the volume of the uniform ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ 15

Q. 4. (a) If PSP′ and QSQ′ are the two perpendicular focal chords of a conic $\frac{1}{r} = 1 + e \cos \theta$, prove that

$$\frac{1}{SP\cdot SP'}+\frac{1}{SQ\cdot SQ'}$$

is constant. 15

(b) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$$ 15

(c) Show that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines, if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$$ 15

(d) If the plane $lx + my + nz = p$ passes through the extremities of three conjugate semidiameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

prove that

$$a^2 l^2 + b^2 m^2 + c^2 n^2 = 3p^2$$ 15

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the family of curves whose tangents form an angle $\frac{\pi}{4}$ with the hyperbolas $xy = c$, $c > 0$. 12

(b) Solve the differential equation

$$\left(xy^2 + e^{-\frac{1}{x^3}}\right) dx - x^2 y\, dy = 0$$ 12

(c) A particle is free to move on a smooth vertical circular wire of radius a. It is projected horizontally from the lowest point with velocity $2\sqrt{ga}$. Show that the reaction between the particle and the wire is zero after is time

$$\sqrt{\frac{a}{g}}\log(\sqrt{5}+\sqrt{6})$$ 12

(d) The middle points of opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T′ be the tensions in these rods, prove that

$$\frac{T}{l}+\frac{T'}{l'}=0$$ 12

(e) Find the depth of the centre of pressure of a triangular lamina with a vertex in the surface of the liquid and other two vertices at depths b and c from the surface. 12

(f) Find the values of constants a, b and c so that the directional derivative of the function.

$$f = axy^2 + byz + cz^2 x^3$$

at the point (1, 2, –1) has maximum magnitude 64 in the direction parallel to z-axis. 12

Q. 6. (a) Solve :

$$(1 + y^2) + (x - e^{-\tan^{-1} y})\frac{dy}{dx} = 0$$ 15

(b) Solve the equiation

$$x^2 p^2 + yp\,(2x + y) + y^2 = 0$$

using the substitution y = u and xy = v and find its singular solution, where

$$p = \frac{dy}{dx}$$ 15

(c) Solve the differential equation

$$x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2\frac{y}{x}=10\left(1+\frac{1}{x^2}\right)$$ 15

(d) Solve the differential equation

$(D^2 - 2D + 2)\, y = e^x \tan x,\ D \equiv \dfrac{d}{dx}$ by the method of variation of parameters. 15

Q. 7. (a) A particle, whose mass is m, is acted upon by a force $m\left(x + \dfrac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a, show that it will arrive at origin in time $\dfrac{\pi}{4}$. 15

(b) If u and V are the velocity of projection and the terminal velocity respectively of a particle rising vertically against a resistance varying as the square of the velocity, prove that the time taken by the particle to reach the highest point is

$$\frac{V}{g}\tan^{-1}\left(\frac{u}{V}\right)$$ 15

(c) Show that the length of an endless chain, which will hang over a circular pulley of radius c so as to be in contact with two-third of the circumference of the pulley is

$$c\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}$$ 15

(d) A uniform rod of length 2a, can turn freely about one end, which is fixed at a height h (<2 a) above the surface of the liquid. If the densities of the rod and liquid be ρ and σ, show that the rod can rest either in a vertical position or inclined at an angle θ to the vertical such that

$$\cos\theta = \frac{h}{2a}\sqrt{\frac{\sigma}{\rho - \sigma}}$$ 15

Q. 8. (a) If $\overline{A} = 2\overline{i} + \overline{k}$, $\overline{B} = \overline{i} + \overline{j} + \overline{k}$, $\overline{C} = 4\overline{i} - 3\overline{j} - 7\overline{k}$, determine a vector $\overline{R}$ satisfying the vector equations.

$$\overline{R} \times \overline{B} = \overline{C} \times \overline{B} \text{ and } \overline{R} \cdot \overline{A} = 0$$ 15

(b) Prove that $r^n \overline{r}$ is an irrotational vector for any value of n, but is solenoidal only if $n + 3 = 0$. 15

(c) If the unit tangent vector $\overline{t}$ and binormal $\overline{b}$ make angles θ and ϕ respectively with a constant unit vector $\overline{a}$, prove that

$$\frac{\sin\theta}{\sin\phi} \cdot \frac{d\theta}{d\phi} = -\frac{k}{\tau}$$ 15

(d) Verify Stokes' theorem for the function

$$\overline{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$ and $y = a$, $a > 0$. 15

SECTION 'A'

Q. 1. Answer any *five* of the following :

(a) Let S be the set of all real numbers except -1. Define $*$ on S by

$$a * b = a + b + ab$$

Is $(S, *)$ a group ?

Find the solution of the equation

$$2 * x * 3 = 7 \text{ in S.}$$ 12

(b) If G is a group of real numbers under addition and N is the subgroup of G consisting of integers, prove that G/N is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication. 12

(c) Examine the convergence of

$$\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$$ 12

(d) Prove that the function f defined by

$$f(x) = \begin{cases} 1, & \text{when x is rational} \\ -1, & \text{when x is irrational} \end{cases}$$

is nowhere continuous. 12

(e) Determine all bilinear transformations which map the half plane $\text{Im}(z) \geq 0$ into the unit circle $|w| \leq 1$. 12

(f) Given the programme

Maximize $u = 5x + 2y$

subject to $x + 3y \leq 12$

$3x - 4y \leq 9$

$7x + 8y \leq 20$

$$x, y \geq 0$$

Write its dual in the standard form. 12

Q. 2. (a) (i) Let O (G) = 108. Show that there exists a normal subgroup or order 27 or 9.

(ii) Let G be the set of all those ordered pairs (a, b) of real numbers for which $a \neq 0$ and define in G, an operation $\otimes$ as follows :

$$(a, b) \otimes (c, d) = (ac, bc + d)$$

Examine whether G is a group w.r.t. the operation $\otimes$. If it is a group, is G abelian ? 10

(b) Show that

$$Z[\sqrt{2}] = \{a + \sqrt{2}\, b \mid a, b \in Z\}$$

is a Euclidean domain. 30

Q. 3. (a) A twice differentiable function f is such that f (a) = f (b) = 0 and f (c) > 0 for a < c < b. Prove that there is at least one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$. 20

(b) Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) > (0, 0) \end{cases}$$

(i) is continuous at (0, 0).

(ii) possesses partial derivatives

$f_x(0, 0)$ and $f_y(0, 0)$. 20

(c) Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Q. 4. (a) With the aid of residues, evaluate

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta, \; -1 < a < 1$$ 15

(b) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. 15

(c) Use the simplex method to solve the problem

$$\begin{aligned} &\text{Maximize } u = 2x + 3y \\ &\text{subject to } -2x + 3y \le 2 \\ &\qquad 3x + 2y \le 5 \\ &\qquad x, y \ge 0 \end{aligned}$$ 30

SECTION 'B'

Q. 5. Answer any *five* of the following :

(a) Solve :

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$$ 12

(b) Solve :

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x\, \partial y^2} = 2\sin(3x + 2y)$$ 12

(c) Evaluate

$$I = \int_0^1 e^{-x^2} dx$$

by the Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2)]$$

$$+ 4f(x_3) + \ldots\ldots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$

with

$2n = 10, \Delta x = 0.1, x_0 = 0, x_1 = 0.1, \ldots, x_{10} = 1.0$ 12

(d) (i) Given the number 59.625 in decimal system. Write its binary equivalent. 6

(ii) Given the number 3898 in decimal system. Write its equivalent in system base 8. 6

(e) Given points A $(0, 0)$ and B (x_0, y_0) not in the same vertical, it is required to find a curve in the $x-y$ plane joining A to B so that a particle starting from rest will traverse from A to B along this curve without friction in the shortest possible time. If $y = y(x)$ is the required curve find the function $f(x, y, z)$ such that the equation of motion can be written as

$$\frac{dx}{dt} = f(x, y(x), y'(x)). \qquad 12$$

(f) A steady inviscid incompressible flow has a velocity field

$$u = fx,\ v = -fy,\ w = 0$$

where f is a constant. Derive an expression for the pressure field $p\{x, y, z\}$ if the pressure

$$p\{0, 0, 0\} = p_0 \text{ and } \vec{g} = -g\,\vec{i}_z. \qquad 12$$

Q. 6. (a) The deflection of a vibrating string of length l, is governed by the partial differential equation $u_{tt} = C^2 u_{xx}$. The ends of the string are fixed at $x = 0$ and l. The initial velocity is zero. The initial displacement is given by

$$u(x, 0) = \frac{x}{l},\ 0 < x < \frac{l}{2}$$

$$= \frac{1}{l}(l - x),\ \frac{l}{2} < x < l.$$

Find the deflection of the string at any instant of time. 30

(b) Find the surface passing through the parabolas $z = 0$, $y^2 = 4ax$ and $z = 1$, $y^2 = -4ax$ and satisfying the equation

$$x\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} = 0 \qquad 15$$

(c) Solve the equation

$$p^2 x + q^2 y = z,\ p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$$

by Charpit's method. 15

Q. 7. (a) If Q is a polynomial with simple roots $\alpha_1, \alpha_2, \ldots \alpha n$ and if P is a polynomial of degree < n, show that

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(x-\alpha_k)}.$$

Hence prove that there exists a unique polynomial of degree < n with given values c_k at the point α_k, k = 1, 2, ... n. 30

(b) Draw a programme outline and a flow chart and also write a programme in BASIC to enable solving the following system of 3 linear equations in 3 unknowns x_1, x_2 and x_3 :

$$C * X = D$$

with

$$C = (c_{ij})^3_{i,j=1}, X = (x_j)^3_{j=1}, D = (d_i)^3_{i=1}.$$ 30

Q. 8. (a) A particle of mass m is constrained to move on the surface of a cylinder. The particle is subjected to a force directed towards the origin and proportional to the distance of the particle from the origin. Construct the Hamiltonian and Hamilton's equations of motion. 30

(b) Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if P be the pressure at the outer surface, the initial pressure at any point on the liquid, distant r from the centre is

$$P \frac{\log r - \log b}{\log a - \log b}$$ 30