C.S.E-Mains 2003

MATHEMATICS

Time Allowed: 3 hours **Maximum Marks: 300**

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER – I

SECTION A

1. Attempt any five of the following:

(a) Let S be any non empty subset of a vector space V over the field F.
Show that the set $\{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n : a_1, a_2, \ldots, a_n \in F, \alpha_1, \alpha_2, \ldots, \alpha_n \in S, n \in N\}$ is the subspace generated by S.

(12)

(b) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, then find the matrix represented by :

$2A^{10} - 10A^{9} + 14A^{8} - 6A^{7} - 3A^{6} + 15A^{5} - 21A^{4} + 9A^{3} + A - I$

(12)

(c) Let f be a real function defined as follows:
$f(x) = x, -1 \le x < 1$
$f(x + 1) = x, \forall x \in R.$
Show that f is discontinuous at every odd integer.

(12)

(d) For all real numbers x, f(x) is given as:

$$f(x) = \begin{cases} e^x + a\sin x, & x < 0 \\ b(x-1)^2 + x - 2, & x \ge 0 \end{cases}$$

Find value a and b for which f is differentiable at a x= 0.

$$f(x) = \begin{cases} e^x + a\sin x, & x < 0 \\ b(x-1)^2 + x - 2 & x \ge 0 \end{cases}$$

Find values of a and b for which f differentiable at x = 0

(12)

(e) A variable plane remains at a constant distance unity from the point (1, 0, 0) and cuts the co-ordinate axes at A, B and C. Find the locus of the centre of the sphere passing through the origin and the point A, B and C.

(12)

(f) Find the equation of the two straight lines through the point (1, 1, 1(that intersect the line x-4 = 4(y -4) = 2(z-1) at an angle of 60^0.

(12)

2. (a) Prove that the eigen vectors corresponding to distinct eigen values of a square matrix are linearly independent.

(12)

(b) If H is a Hermitian matrix, then show that $A=(H+iI)^{-1}$ H-iI) is a unitary matrix. Also show that every unitary matrix can be expressed in this form, provided 1 is not an eigenvalue of A

(15)

(c) If $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ then find a diagonal matrix D and a matrix B such that A=BDB' where B' denotes the transpose of B.

(15)

(d) Reduce the quadratic form given below to canonical form and find its rank and signature:
$x^2 + 4y^2 + 9z^2 + u^2 - 12yx + 6zx - 4xy - 2xu - 6zu$.

(15)

3. (a) A rectangular box, open at the top, is to have a volume of $4m^3$ using Lagrange's method of multipliers, find the dimensions of the box so that the martial of a given type required to construct it may be least.

(15)

(b) Test the convergence of the integral:

(i) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

(ii) $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$

(15)

(c) Evaluate the integral $\int_0^a \int_{y^2/a}^{a} \frac{y\, dx\, dy}{(a-x)\sqrt{(ax-y^2)}}$

(15)

(d) Find the volume generated by resolving the area, bounded by the curves $(x^2 + 4a^2)y = 8a^3$, $2y = x$, and $x = 0$, about the y-axis.

(15)

4. (a) Find the volume of the tetrahedron formed by the four planes $lx + my + nz = p$, $lx + my = 0$, $my + nz = 0$, and $nz + lx = 0$

(15)

(b) A sphere of constant radius r passes through the origin O and cuts the co-ordinate axes at A, B and C. Find the locus of the foot of the perpendicular from O to the plane ABC.

(15)

(c) Find the equations of the lines of intersection of the plane $x + 7y - 5z$ and the cone $3xy + 14zx - 30xy = 0$.

(15)

(d) Find the equations of the line of shortest distance between the lines:
$y + z = 1, x = 0$ and $x - z = 1, y = 0$ as the intersection of two planes.

(15)

SECTION B

5. Attempt any five out of the following:

(a) Show that the orthogonal trajectory of a system of focal ellipses is self orthogonal.

(12)

(b) Solve: $x\frac{dy}{dx} + y \log y = xye^x$

(15)

(c) A sphere of weight W and radius a lies within a fixed spherical shell of radius b. A particle of weight w is fixed to the upper end of the vertical diameter. Prove that the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$

(12)

(d) A particle describes the curve
$r = a(1 + \cosh\theta) / (\cosh\theta - 2)$
under a force F to the pole. Show that the law of force is $F \propto 1/r^4$.

(12)

(e) Show that if a', b' and c' are the reciprocals to the non-coplanar vectors a, b and c, then any vector r may be expressed as $r = (r.a')a + (r.b')b + (r.c')c$

(12)

(f) Prove that the divergence of a vector field is invariant w.r. to co-ordinate transformations.

(12)

6. (a) Solve $(D^5-D)y = 4(e^x + \cos x + x^3)$, where $D = d/dx$.

(15)

(b) Solve the differential equation $(px^2 + y^2)(px + y) = (p + 1)^2$, where $p = dy/dx$, by reducing it to Clairaut's form using suitable substitutions.

(15)

(c) Solve: $(1 + x)^2 y'' + (1 + x)y' + y = \sin 2[\log(1 + x)]$

(15)

(d) Solve the differential equation
$x^2y'' - 4xy' + 6y = x^4 \sec^2 x$
by variation of parameters.

(15)

7. (a) An elastic string of natural length $a + b$, where $a > b$, and modulus of elasticity λ, has a particle of mass m attached to it at a distance a from one end which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, find the periodic time and the distance in which the particle will oscillate to and fro.

(15)

(b) If a particle slides down a smooth cycloid, starting from a point whose arcual distance from the vertex is b, prove that its speed at any time t is $2\pi b/T \sin(2\pi t/T)$, where T is the time of complete oscillation of the particle.

(15)

(c) A ladder on a horizontal floor leans against a vertical wall. The coefficients of friction of the floor and the wall with the ladder are μ and μ_1 respectively. If a man, whose weight is n times that of the ladder, want to climb up the ladder, find the minimum safe angle of the ladder with the floor.

(15)

(d) An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is immersed vertically in a fluid with its semi-axis of length a horizontal. If its centre be at a depth h, find the depth of the centre of pressure.

(15)

8. (a) Let the position vector of a particle moving on a plane curve be r(t), where t is the time. Find the components of its acceleration along the radial and transverse directions.

(15)

(b) Prove that identity $\nabla A^2 = 2(A.\nabla)A + 2A\times(\nabla\times A)$ where $\nabla = \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

(15)

(c) Find the radii of curvature and torsion at a point of intersection of the surfaces $x^2-y^2=c^2$, $y = x \tanh(z/c)$.

(15)

(d) Evaluate $\iint_S curl\,A.dS$, where S is the open surface $x^2+y^2-4x+4z=0$, $z\geq 0$ and

$A=(y^2+z^2-x^2)\hat{i}+(2z^2+x^2-y^2)\hat{j}+(x^2+y^2-3z^2)\hat{k}$

(15)

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MATHEMATICS

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PAPER - II

SECTION A

1. Answer any five of the following:

(a) If H is a subgroup of a group G s.t. $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G.

(12)

(b) Show that the ring $Z[i] = (a+bi) | a \in Z, b \in Z, i = \sqrt{-1})$ of Gaussian integers is a Euclidean domain.

(12)

(c) Let a be a positive real number and (x_n) a sequence of rational numbers such that $\lim_{n\to\infty} x_n = 0$. Show that $\lim_{n\to\infty} a^{x_n} = 1$.

(12)

(d) If a continuous function of x satisfies the functional equation $f(x + y) = f(x) + f(y)$ then show that $f(x) \alpha x$ where α is a constant.

(12)

(e) Determine all the bilinear transformations which transform the unit circle $|z| \le 1$ into the unit circle $|w| \le 1$

(12)

(f) For the following system of equations:

$x_1 + x_2 + x_3 = 3$

$2x_1 - x_2 + 3x_3 = 4$

Determine:

(i) all basic solutions

(ii) all basic feasible solutions

(iii) a feasible solutions which is not a basic feasible solutions

2. (a) (i) Let R be the ring of all real-valued continuous functions on the closed interval [0, 1]. Let $M = \{f(x) \in R | f(1/3) = 0\}$. Show that M is a maximal ideal of R.

(10)

(ii) Let M and N be two ideals of a ring R. Show that $M \cup N$ is an ideal of R if and only if either $M \subseteq N$ or $N \subseteq M$.

(10)

(b) (i) Show that $Q(\sqrt{3}, i)$ is a splitting field for $x^5 - 3x^3 + x^2 - 3$ where Q is the field of rational numbers.

(15)

(ii) Prove that $x^2 + x + 4$ is irreducible over F the field of integers module 11 and prove further that $\frac{F[x]}{(x^2 + x + 4)}$ is a field having 121 elements.

(15)

(c) Let R be a unique factorization domain (UFD), then prove that R[x] is also UFD.

(10)

3. (a) Show that the maximum value of $x^2y^2z^2$ subject to the condition $x^2 + y^2 + z^2 = c^2$ is $c^6/27$. Interpret the result.

(20)

(b) The axes of two equal cylinders intersect at right angles. If a be their radius, then find the volume common to the cylinders by the method of multiple integrals.

(20)

(c) Show that $\int_0^\infty \frac{dx}{1+x^2\sin^2 x}$ is divergent.

(20)

4. (a) (i) Discuss the transformation $W = \left(\frac{z-ic}{z+ic}\right)^2$ (c real) showing that the upper half of the W-plane corresponds to the interior of the semicircle lying to the right of imaginary axis in the z-plane.

(15)

(ii) Using the method of contour integration to prove that $\int_0^\pi \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ $(a > 0)$

(15)

(b) (i) An animal feed company must produce 200 kg of a mixture consisting of ingredients X_1 and X_2 daily. X_1 costs Rs. 3 per kg and X_2 costs Rs. 8 per kg. No more than 80 kg of X_1 can be used, and at least 60 kg of X_2 must be used. Formulate a linear programming model of the problem and use Simplex Method to determine the ingredients X_1 and X_2 to be used to minimize costs.

(15)

(ii) Find the optimal solution for the assignment problem with the following cost matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 6 & 1 & 9 & 11 & 12 \\ 2 & 8 & 17 & 2 & 5 \\ 11 & 8 & 3 & 3 & 3 \\ 4 & 10 & 8 & 6 & 11 \\ 8 & 10 & 11 & 5 & 13 \end{bmatrix} \end{array}$$

Indicate clearly the rule you apply to arrive at the complete assignment.

(15)

SECTION B

5. Answer any five of the following:

(a) Find the general solution of $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y + \cos(2x + 3y)$

(12)

(b) Show that the differential equation of all cones which have their vertex at the origin are $px + qy = z$. Verify that $yx + zx + xy = 0$ is a surface satisfying the above equation.

(12)

(c) Evaluate $\int_0^\infty e^{-x}dx$; employing three points

Gaussian quadrature formula, finding the required weights and residues. Use five decimal places for computation.

(12)

(d) (i) Convert the following binary number into octal and hexa decimal system:
101110010.10010

(6)

(ii) Find the multiplication of the following binary numbers:
11001.1 and 101.1

(6)

(e) A solid body of density ρ is in the shape of the solid formed by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line. Show that its moment of inertia about the straight line through the pole and perpendicular to the initial line is $\frac{352}{105}\pi \rho a^5$.

(12)

(f) For an incompressible homogenous fluid at the point (x, y, z) the velocity distribution is given by $u = -\frac{c^2 y}{r^2}, v = \frac{c^2 x}{r^2}, w = 0$ where r denotes the distance from z-axis.

Show that it is a possible motion and determine the surface which is orthogonal to stream line.

(12)

6. (a) (i) Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = xy + e^{x+2y}$

(15)

(ii) Solve the equation

$p^2 - q^2 - 2pq - 2qy + 2xy = 0$

Using Charpti's method. Also find the singular solution of the equation, if it exists.

(15)

(b) Find the deflection u(x, t) of a vibrating string, stretched between fixed points (0, 0) and (3*l*, 0), corresponding to zero initial velocity and following initial deflection.

$$f(x) = \begin{cases} \frac{hx}{l} & \text{when } 0 \le x \le l \\ \frac{h(3l-2x)}{l} & \text{when } l \le x \le 2l \\ \frac{h(x-3l)}{l} & \text{when } 2l \le x \le 3l \end{cases}$$

where h is a constant.

(30)

7. (a) Find the positive root of the equation

$2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$

Using Newton-Raphson method correct to four decimal places. Also show that the following scheme for error of second order:

$x_{n+1} = \frac{1}{2}X_n\left(1 + \frac{a}{x_n^2}\right)$

(30)

(b) Draw a flow chart and write a programme BASIC for Simpson's $1/3^{rd}$ rule for integration $\int_{2}^{3} \frac{1}{1+x^2}dx$ correct at 10^{-6}.

(30)

8. (a) A line circular tube, radius c, lies on a smooth horizontal plane, and contains two equal particles connected by an elastic string in the tube, the natural length of which is equal to half the circumference. The particles are in contact and fastened together, the string being stretched round the tube.

If the particle become disunited, prove that the velocity of the tube when the string has regained its natural length is

$$\left\{\frac{2\pi\lambda mc}{M(M+2m)}\right\}^{1/2}$$

When M, m are the masses of the tube and each particle respectively, and λ is the modulus of elasticity.

(30)

(b) (i) Two sources, each of strength m are placed at the points (-a, 0) and (a, 0) and a sink of strength 2m is placed at the origin. Show that the stream lines are the curves $(x^2 + y^2)^2 = (x^2 - y^2 + \lambda xy)$ where λ is a variable parameter.

Also show that the fluid speed at any point is $\dfrac{2ma^2}{r_1 r_2 r_3}$, where r_1, r_2 and r_3 are respectively the distances of the point from the sources and sink.

(15)

(ii) An infinite mass of fluid is acted upon by a force $\mu r^{-3/2}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of a sphere $r = c$ in it; show that the cavity will be filled up after an interval of the $\{2/5\mu\}^{1/2} c^{5/4}$.

(15)