C.S.E-Mains 2002

MATHEMATICS

Time Allowed: 3 hours *Maximum Marks: 300*

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER – I

SECTION-A

1. Attempt any five of the following:

(a) Show that the mapping $T: R^3 \to R^3$ where T(a, b, c) = (a - b, b - c, a + c) is linear and non singular.

(12)

(b) A square matrix A is non singular if and only if the constant term in its characteristic polynomial is different from zero.

(12)

(c) Show that $\frac{b-a}{\sqrt{1-a^2}} \le \sin^{-1} b - \sin^{-1} a \le \frac{b-a}{\sqrt{1-b^2}}$ for 0 < a < b < 1.

(12)

(d) Show that $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy = \pi / 4$.

(12)

(e) Show that the equation
$9x^2 - 16y^2 - 18x - 32y - 151 = 0$
represents a hyperbola. Obtain its eccentricity and foci.

(12)

(f) Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes
x = 0, y = 0, z = 0 and x + y + z = a.

(12)

2. (a) Let $T: R^5 \to R^4$ be a linear mapping given by T(a, b, c, d, e) = (b - d, d + e, b, 2d + c, b + e). Obtain bases for its null space and range space.

(15)

(b) Let A be a real 3 × 3 symmetric matrix with eigen values 0, 0 and 5. If the corresponding eigen vectors are (2, 0, 1), (2, 1, 1) and (1, 0, -2), then find the matrix A.

(15)

(c) Solve the following system of linear equations:
$x_1 - 2x_2 - 3x_3 + 4x_4 = -1$
$-x_1 + 3x_2 + 5x^3 - 5x_4 - 2x = 0$
$2x_1 + x - 2x_3 + 3x_4 - 4x_5 = 17$

(15)

(d) Use Cayley- Hamilton theorem to find the inverse of the following matrix:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

(15)

3. (a) Let $f(x)=\begin{cases} x^p \sin 1/x, & x \neq 0 \\ 0 & , x=0 \end{cases}$ Obtain condition on p such that (i) f is continuous at x = 0 and (ii) f is differentiable at x = 0.

(15)

(b) Consider the set of triangles having a given base and a given vertex angle. Show that the triangle having the maximum area will be isosceles.

(15)

(c) if the roots of the equation

$(\lambda - u)^3 + (\lambda - v)^3 + (\lambda - w)^3 = 0$

in λ are x, y, z, show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{2(u-v)(v-w)(w-u)}{(x-y)(y-z)(z-x)}$$

(15)

(d) Find the centre of gravity of the region bounded by the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$ and both axes in the first quadrant, the density benign $\rho = kxy$, where k is constant.

(15)

4. (a) Tangents are drawn from any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the circle $x^2+y^2=r^2$. Show that the chords of contact are tangents to the ellipse $a^2x^2+b^2y^2=r^4$.

(15)

(b) Consider a rectangular parallelepiped with edges a, b, c. Obtain the shortest distance between one of its diagonals and an edge which does not intersect this diagonal.

(15)

(c) Show that the feed of the six normals drawn from any point (α, β, γ) to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie on the cone

$$\frac{a^2\left(b^2-c^2\right)\alpha}{x}+\frac{b^2\left(c^2-a^2\right)\beta}{y}+\frac{c^2\left(a^2-b^2\right)\gamma}{z}=0$$

(15)

(d) A variable plane parallel to the plane

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ meets the co-ordinate axes of A, B and C. Show that the circle ABC lies on the conic

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

(15)

SECTION - B

5. Attempt any five of the following:

(a) Solve: $x\frac{dy}{dx}+3y=x^3y^2$

(12)

(b) Find the value of λ for which all solution of

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-\lambda y=0$$

tend to zero as $x \to \infty$.

(12)

(c) A particle of mass m is acted upon by a force $m\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a from the origin, show that the time taken by it to reach the origin is $\pi/4$.

(12)

(d) Obtain the equation of the curve in which a string hangs under gravity from two fixed points (not lying in a vertical line), when line mass density at each of its points varies as the radius of curvature of the curve.

(12)

(e) Half of the ellipse is vertically immersed in water with minor axis just in the surface. Find the position of centre of pressure.

(12)

(f) Let $\bar{R}$ be the unit vector along the vector $\bar{r}(t)$. Show that $\bar{R}\times\frac{d\bar{R}}{dt}=\frac{\bar{r}}{r^2}\times\frac{d\bar{r}}{dt}$ where $r=|\bar{r}|$.

(12)

6. (a) Find the value of constant λ such that the following differential equation becomes exact.
$\left(2xe^y+3y^2\right)dy/dx+\left(3x^2+\lambda e^y\right)=0.$
Further, for this value of λ, solve the equation.

(15)

(b) Solve: $\frac{dy}{dx}=\frac{x+y+4}{x-y-6}$

(15)

(c) Using the method of variation of parameters, find the solution of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$ with y(0) = 0 and $(dy/dx)_{x=0}=0$..

(15)

(d) Solve: $(D-1)(D^2-2D+2)y=e^x$ where $D=d/dx$.

(15)

7. (a) A heavy particle of mass m slides on a smooth arc of a cycloid in a medium whose resistance in $mv^2/2c$, v being the velocity of the particle and c being the distance of the starting point from the vertex. If the axis is vertical and vertex upwards, find the velocity of the particle at the cusp.

(15)

(b) A particle describes a curve with constant velocity and its angular velocity about a given point O varies inversely as its distance from O. Show that the curve is an equiangular spiral.

(15)

(c) Five weightless rods of equal lengths are jointed together so as to form a rhombus ABCD with a diagonal BD. If a weight W be attached to C and the system be suspended from a point A, show that the thrust BD is equal to $W/\sqrt{3}$.

(15)

(d) A solid cylinder floats in a liquid with its axis vertical. Let σ be the ratio of the specific gravity of the cylinder to that of the liquid. Prove that the equilibrium is stable if the ratio of the radius of the base to the height is greater than $\sqrt{2\sigma(1-\sigma)}$.

(15)

8. (a) Find the curvature k for the space curve:
$x=a\cos\theta,\ y=a\sin\theta,\ z=a\theta\tan\alpha$

(15)

(b) Show that $curl\left(curl\,\bar{v}\right)=grad\left(div\,\bar{v}\right)-\nabla^2\bar{v}$

(15)

(c) Let D be a closed and bounded region having boundary S. Further, let f be a scalar function having second order partial derivatives defines on it. Show that

$\iint_S (f \text{ grad } f).\hat{n}\, ds = \iiint_V \left[|\text{grad } f|^2 + f\nabla^2 f \right] dV$. Hence or otherwise evaluate $\iint_S (f \text{ grad } f).\hat{n}\, dS$ for f = 2x + y + 2z over S = $x^2 + y^2 + z^2 = 4$.

(15)

(d) Find the value of constants a, b and c such that the maximum value of directional derivative of f = $axy^2 + byz + cx^2x^2$ at (1, -1, 1) is in the direction parallel to y-axis and has magnitude 6.

(15)

MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER - II

SECTION A

1. Attempt any five of the following:

(a) Show that a group of order 35 is cyclic.

(12)

(b) Show that polynomial $25x^4 + 9x^3 + 3x + 3$ is irreducible over the field of rational numbers.

(12)

(c) Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if m > 0.

(12)

(d) Find all the positive values of a for which the series $\sum_{n=1}^{\infty}\frac{(an)^n}{n!}$ converges.

(12)

(e) Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with $f\left(\frac{1}{n}\right)=g\left(\frac{1}{n}\right)$ for n = 1, 2, 3....... Then show that f(z) = g(z) for each z in $\mathbb{C}$.

(12)

(f) Using Simplex method
Maximize $45x_1 + 80x_2$
Subject to
$5x_1 + 20x_2 \leq 400,$
$10x_1 + 15x_2 \leq 450,$
$x_1, x_2 \geq 0.$

(12)

2. (a) (i) Show that a group of p^2 is abelian, where p is a prime number.

(10)

(ii) Prove that a group of order 42 has a normal subgroup of order 7.

(10)

(b) Prove that in the ring F[x] of polynomial over a field F, the ideal I = [p(x)] is maximal if and only if the polynomial p(x) is irreducible over F.

(20)

(c) (i) Show that every finite integral domain is a field.

(10)

(ii) Let F be a field with q elements. Let E be a finite extension of degree n over F. Show that E has q^n elements.

(10)

3. (a) Test uniform convergence of the series
$\sum_{n=1}^{\infty}\frac{\sin nx}{n^p}$ where p > 0.

(20)

(b) Obtain the maxima and minima of
$x^2 + y^2 + z^2 - yz - xz - xy$
subject to the condition
$x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0.$

(25)

(c) A solid hemisphere H of radius 'a' has density ρ depending on the distance R from the centre and is given by:
$\rho = k(2a - R)$
where k is a constant.
Find the mass of the hemisphere, by the method of multiple integrals.

(15)

4. (a) (i) Show that, when $0 < |z-1| < 2$, then function $f(x) = \frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in power of (z-1) as $\frac{-1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$

(15)

(ii) Establish, by contour integration, $\int_0^{\infty}\frac{\cos(ax)}{x^2+1}dx = \frac{\pi}{2}e^{-a}$ where $a \ge 0$.

(15)

(b) (i) Using Simplex method maximize

$Z = 5x_1 + 3x_2$ Subject to
$x_1 + x_2 \le 2,$ $5x_1 + 2x_2 \le 10,$
$3x_1 + 8x_2 \le 12,$ $x_1, x_2 \ge 2.$

(15)

(ii) A company has 3 factories A, B and C which supply units to warehouses X, Y and Z. Every month the capacities of the factories per month are 60, 70 and 80 units at A, B and C respectively. The requirements of X, Y and Z per month are 50, 80 and 80 respectively. The necessary data in terms of unit transportation costs in rupees, factory capacities and warehouse requirements are given below:

	X	Y	Z	
A	8	7	5	60
B	6	8	9	70
C	9	6	5	80
	50	80	80	210

Find the minimum distribution cost.

(15)

SECTION - B

5. Attempt any five of the following:

(a) Find two complete integrals of the partial differential equation $x^2p^2 + y^2q^2 - 4 = 0$

(12)

(b) Find the solution of the equation $z = 1/2\,(p^2 + q^2) + (p - x)(q - y)$.

(12)

(c) Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by the method of false position.

(12)

(d) (i) Convert $(100.85)_{10}$ into its binary equivalent.

(4)

(ii) Multiply the binary numbers $(1111.01)_2$ and $(1101.11)_2$ and check with its decimal equivalent.

(8)

(e) Find the moment of inertia of a circular wire about

(i) a diameter; and

(4)

(ii) a line through the centre and perpendicular to its plane.

(12)

(f) Show that the velocity potential $\phi = 1/2\, a(x^2 + y^2 - 2z^2)$ satisfied the Laplacae equation, and determine the stream lines.

(12)

6. (a) Frame the partial differential equation by eliminating the arbitrary constants a and b from $\log(az - 1) = x + ay + b$.

(10)

(b) Find the characteristic strip of the equation

$xp + yq - pq = 0$

and then find the equation of the integral surface through the curve $z = x/2,\ y = 0$.

(20)

(c) Solve: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < l,\ t > 0$

$u(0, t) = u(l, t) = 0$

$u(x, 0) = x(l-x),\ 0 \le x \le l.$

(30)

7. (a) (i) Find the cubic polynomial which takes the following values:

$Y(0) = 1,\ y(1) = 0,\ y(2) = 1$ and $y(3) = 10$.

Hence, or otherwise, obtain y(4).

(10)

(ii) Given $dy/dx = y - x$ where $y(0) = 2$, using the Runge-Kutta fourth order method, find y(0, 1) and y(0, 2). Compare the approximate solution with its exact solution. ($e^{0.1} = 1.10517,\ e^{0.2} = 1.2214$).

(20)

(b) (i) A teacher conducts there tests TEST 1, TEST 2, and FINAL for 50 marks each. Out of the marks scored in the two tests, TEST 1 and TEST 2, he takes the betterone and adds to the marks scored in FINAL, so that the total marks scored will be for a maximum 100.

The letter grades will be assigned depending on the marks scored as per the following norm:

0 – 39 : E
40 – 49 : D
50 – 59 : C
60 – 74 : B
75 – 100 : A.

For each student data consisting of name, scores in TEST 1, TEST 2, and FINAL are given. Write a program in BASIC which will print out the names, total marks scored and grade obtained for all 20 students in a class.

(20)

(ii) Draw a flow chart to examine whether a given number is a prime.

(10)

8. (a) A thin circular disc of mass M and radius a can turn feely about a thin axis OA, which is perpendicular to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity w about its end A.

Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}(g/aw^2)$ unless $w^2 < g/a$ and then θ is zero.

(30)

(b) (i) Show that: $u = \frac{-2xyz}{\left(x^2+y^2\right)^2}, v = \frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}, w = \frac{y}{x^2+y^2}$

are the velocity components of a possible liquid motion. Is this motion irrigational?

(ii) Prove that: $\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial(x, y)}$

Where v is the kinematic viscosity of the fluid and ψ is the stream function for a two-dimensional motion of a viscous fluid.

(15)