C.S.E-Mains 2001

MATHEMATICS

Time Allowed: 3 hours *Maximum Marks: 300*

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER–I

SECTION-A

1. Attempt any five of the following:

(a) Show that the vector (1, 0, -1), (0, -3, 2) and (1, 2, 1) form a basis for the vector space $R^3(R)$. (12)

(b) If λ is a characteristic root of a non-singular matrix A, then prove that |A|/λ is a characteristic root of Adj. A (12)

(c) Let f be defined on IR by setting f(x) = x, if x is rational, and f(x) = 1-x, if x is irrational. Show that f is continuous at x = 1/2, but is continuous at every other point. (12)

(d) Test the convergence of $\int_0^1 \frac{\sin(1/x)}{\sqrt{x}} dx$ (12)

(e) Show that the equation

$x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$

represents a hyperbola. Find the coordinates of its centre and the length of its real semi-axes. (12)

(f) Find the shortest distance between the axis of z and the line

ax + by + cz + d = 0, a′x + b′y + c′z + d′ = 0 (12)

2. (a) If

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Show that for every integer $n \geq 3$,

$A^n = A^{n-2} + A^2 - I$

Hence, determine A^{50} (15)

(b) When is a square matrix A said to be congruent to a square matrix B? Prove that every matrix congruent to a skew-symmetric matrix is skew-symmetric. (15)

(c) Determine an orthogonal matrix P such that P-1 AP is a diagonal matrix, where

$$A = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$$

(15)

(d) Show that the real quadratic from

$$\phi = n\left(x_1^2 + x_2^2 + ... + x_n^2\right) - \left(x_1 + x_2 + ... + x_n\right)^2$$

In n variable is positive semi-definite.

(15)

3. (a) Find the equation of the cubic curve which has the same asymptotes as

$$2x(y-3)^2 = 3y(x-1)^2$$

And which touches the x-axis at the origin and passes through the point (1, 1)

(15)

(b) Find the maximum and minimum radii vectors of the section of the surface

$$\left(x^2 + y^2 + z^2\right)^2 = a^2x^2 + b^2y^2 + c^2z^2$$

By the plane $lx + my + nz = 0$

(15)

(c) Evaluate

$$\iiint (x + y + z + 1)^2\, dxdydz$$

Over the region defined by

$$x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$$

(15)

(d) Find the volume of the solid generated by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line.

(15)

4. (a) Find the equation of the circle circumscribing the triangle formed by the point $(a,0,0),(0,b,0),(0,0,c)$. Obtain also the coordinates of the centre of the circle.

(15)

(b) Find the locus of equal conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(15)

(c) Prove that

$$5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$$

Represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$.

(15)

(d) If TP, TQ, and $T'P', T'Q'$ all lie on a conic.

(15)

SECTION - B

5. Attempt any five of the following:

(a) A continuous function y(t) satisfies the differential equation

$$dy/dt = \begin{cases} 1 + e^{1-t}, & 0 \leq t < 1 \\ 2 + 2t - 3t^2, & 1 \leq t < 5 \end{cases}$$

If y(0) = -e, find y(2).

(12)

(b) Solve:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$

(12)

(c) Find the law of force to the pole when the path of a particle is the cardioid r = a(1 − cos θ) and prove that if F be the force at the apse and υ the velocity there, then $3\upsilon^2 = 4aF$.

(12)

(d) The middle points of the opposite sides of a jointed quadrilateral are connected by light rods of lengths, l, l'. If T, T' be the tensions is these rods, prove that

$$\frac{T}{l}+\frac{T'}{l'}=0$$

(12)

(e) A solid right circular cone with semi-vertical angle α is just immersed in a liquid with a generating line on the surface. If θ be the inclination of the vertical with the resultant thrust on the curved surface, prove that

$$\left(1-3\sin^2\alpha\right)\tan\theta=3\sin\alpha\cos\alpha$$

(12)

(f) Find the length of the arc of the twisted curve $r=\left(3t,3t^2,2t^3\right)$ from the point t = 0 to the point t = 1. Find also the unit tangent t, unit normal n and the unit binomial b at t = 1.

(12)

6. (a) Solve:

$$\frac{dy}{dx}+\frac{y}{x}\log_e y=\frac{y(\log_e y)^2}{x^2}$$

(15)

(b) Find the general solution of

$$ayp^2+(2x-b)p-y=0, a>0$$

(15)

(c) Solve:

$$\left(D^2+1\right)^2 y=24x\cos x$$

Given that $y=Dy=D^2y=0$ and $D^3y=12$ when x = 0.

(15)

(d) Using the method of variation of parameters, solve

$$\frac{d^2y}{dx^2}+4y=4\tan 2x$$

(15)

7. (a) A comet describing a parabola under inverse square law about the sun, when nearest to it suddenly breaks up, with out gain or loss of kinetic energy, into two equal portions, one of which describes a circle. Prove that the other will describe a hyperbola of eccentricity 2.

(15)

(b) A particle a mass M is at rest and begins to move under the action of a constant force F in a fixed direction. If encounters the resistance of a stream of fine dust moving in the opposite direction with velocity V, which deposits matter on it at a constant rate ρ. Show that the mass of the particle will be m when it has travelled a distance

$$\frac{k}{\rho^2}\left[m-M\left\{1+\log\frac{m}{M}\right\}\right]$$

Where k = F - ρV.

(15)

(c) OA, OB and OC are edges of a cube of side a, and OO' AA', BB' and CC' are its diagonals. Along OB', O'A, BC and C'A' act forces equal to P, 2P, 3P and 4P respectively. Reduce the system to a force at O together with a couple.

(15)

(d) A right circular cylinder floating with its axis horizontal and in the surface is displaced in the vertical plane through the axis. Discuss its stability of equilibrium.

(15)

8. (a) Show that

$$curl\frac{a \times r}{r^3} = -\frac{a}{r^3} + \frac{3r}{r^5}(a.r)$$

Where a is a constant vector.

(15)

(b) Find the directional derivative of $f = x^2yz^3$ along $x = e^{-t}, y = 1 + 2\sin t, z = t - \cos t$ at $t = 0$

(15)

(c) Show that the vector field defined by

$F = 2xyz^3 i + x^2 z^3 j + 3x^2 yz^2 k$

is irrotational. Find also the scalar u such the F = grad u.

(15)

MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER - II
SECTION A

1. Attempt any five of the following:

(a) Let K be a field and G be a finite subgroup of the multiplicative group of nonzero elements of G. Show that G is a cyclic group.

(12)

(b) Prove that the polynomial $1 + x + x^2 + ... + x^{p-1}$, where p is a prime number, is irreducible over the field of rational numbers.

(12)

(c) Show that $\int_0^{\pi/2} \frac{x^n}{\sin^m x} dx$ exists if and only if m < n + 1.

(12)

(d) If $\underset{n \to \infty}{Lt}\, a_n = l$, then prove that $\underset{n \to \infty}{Lt} \frac{a_1 + a_2 + ... + a_n}{n} = l$.

(12)

(e) Prove that the Riemann zeta function ξ defined by $\xi(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for Re z > 1 and converges uniformly Re $z \geq 1 + \varepsilon$ where $\varepsilon > 0$ is arbitrary small.

(12)

(f) Compute all basic feasible solutions of the linear programming problem

$Max \quad z = 2x_1 + 3x_2 + 2x_3$

$subject\ to\ 2x_1 + 3x_2 - x_3 = 8$

$x_1 - 2x_2 + 6x_2 + 6x_3 = -3$

$x_1, x_2, x_3 \geq 0.$

And hence indicate the optimal solution.

(12)

2. (a) Let N be a normal subgroup of a group G. Show that G/N is abelian if and only if for all $x, y \in G, xyz^{-1}y^{-1} \in N$.

(20)

(b) If R is a commutative ring with unit element and M is an ideal of R, then show that M is a maximal ideal of R if and only if R/M is a field.

(20)

(c) Prove that every finite extension of a field is an algebraic extension. Given an example to show that the converge is not true.

(20)

3. (a) A function f is defined in the interval (a, b) as follows:

$f(x) = 1/q^2, when\ x = p/q$

$= 1/q^3, when\ x = \sqrt{p/q}$

Where p, q are relatively prime integers.

$f(x) = 0$ for all other values of x.

Is f Riemann integrable? Justify your answer.

(20)

(b) Show that U = xy + yz + zx has a maximum value when the three variables are connected by the relation ax + by + cz = 1 and a, b, c are positive constant satisfying the condition 2(ab + bc + ca) > (a^2 + b^2 + c^2)

(25)

(c) Evaluate

$\iiint (ax^2 + by^2 + cz^2)\,dxdydz$ taken through out the region $x^2 + y^2 + z^2 \le R^2$.

(15)

4. (a) (i) Find the Laurent series for the function $e^{1/z}$ in $0 < z < \infty$. Using this expansion, show that $1/\pi \int_0^{\pi} \exp(\cos\theta)\cos(\sin\theta - n\theta)\,d\theta = 1/n!$

(ii) Show that $\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$.

(15)

(b) (i) Using duality or otherwise solve the liner programming problem

Minimize $18x_1 + 12x_2$

Subject to $2x_1 - 2x_2 \ge -3$

$3x_1 + 2x_2 \ge 3$

$x_1, x_2 \ge 0$

(15)

(ii) A manufacturer has distribution centers at Delhi, Kolkata and Chennai. These centers have available 30, 50, and 70 units of his product. His four retail outlets require the following number of units: A, 30; B, 20; C, 60; D, 40;

The transportation cost per unit in rupees between each centre and outlet is given in the following table:

Distribution Centres	Retail outlets			
	A	B	C	D
Delhi	10	7	3	6
Kolkata	1	6	7	3
Chennai	7	4	5	3

Determine the minimum transportation cost.

(15)

SECTION - B

5. Attempt any five of the following:

(a) Find the complete integral of the partial differential equation

$2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2 + y^2)$

(12)

(b) Find the general integral of the equation

$\{my(x+y) - nz^2\}\partial z/\partial x - \{lx(x+y) - nz^2\}\partial z/\partial y = (lx - my)z.$

(12)

(c) Show that the truncation error associated with linear interpolation of f(x), using ordinates at x_0 and x_1 with $x_0 \le x \le x_1$ is not larger in magnitude then $1/8M_2(x_1 - x_0)^2$ where $M_2 = \max|f''(x)| \; in \; x_0 \le x \le x \le x_1$.

Hence show that if $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

The truncation error corresponding to linear interpolation of f(x) in $x_0 \le x \le x_1$ cannot exceed $\frac{(x_1 - x_0)^2}{2\sqrt{2\pi e}}$.

(12)

(d) (i) Given $A.B' + A'.B = C$

Show that $A.C' + A'.C = B$.

(ii) Express the area of the triangle having sides of lengths $6\sqrt{2}, 12, 6\sqrt{2}$ units in binary number system.

(6 + 6 = 12)

(e) Determine the moment of inertial of a uniform hemisphere about its axis of symmetry and about an axis perpendicular to the axis of symmetry and through centre of the base.

(12)

(f) If the velocity distribution of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{kz^2 - r^2}{r^5}\right)$ then determine the parameter k such that it is a possible motion.

Hence find its velocity potential.

(12)

6. (a) Prove that for the equation

$z + px + qy - 1 - pqx^2y^2 = 0$

the characteristic strips are given by

$$x(t) = \frac{1}{B + Ce^{-t}}, y(t)\frac{1}{A + De^{-t}},$$

$$z(t) = E - (AC + BD)e^{-t}$$

$$p(t) = A\left(B + Ce^{-t}\right)^2, q(t) = B\left(A + De^{-t}\right)^2$$

Where A, B, C, D and E are arbitrary constant. Hence find the values of these arbitrary constants if the integral surface passes through the line z = 0, x = y.

(30)

(b) (i) Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by

$$x\left(x^2 + y^2 + z^2\right) = C_1 y^2.$$

(10)

(ii) Solve the equation

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x^2 y^4$$

by reducing it to the equation with constant coefficients.

(20)

7. (a) using Gauss Seidel iterative method and the starting solution $x_1 = x_2 = x_3 = 0$, determine the solution of the following system of equation in two iterations

$10x_1 - x_2 - x_3 = 0$

$x_1 + 10x_2 + x_3 = 12$

$x_1 - x_2 + 10x_3 = 10$

Compare the approximate solution with the exact solution.

(30)

(b) (i) Write a computer program in BASIC to evaluate the polynomial

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$

for values of x = 1.2(0.2)2.0.

(15)

(ii) Find the values of the two-valued variables A, B, C and D by solving the set of simultaneous equations

A' + A . B = 0

A . B = A . C

A . B + A . C' + C . D = C' . D

(15)

8. (a) Find the equation of motion for a particle of mass m which is constrained to move on the surface of a cone of semi-vertical angle α and which is subjected to a gravitational force.

(30)

(b) Show that the velocity distribution in axial flow of viscous incompressible fluid along a pipe of annular cross-section, radii $r_1 < r_2$, is given by

$$\omega = \frac{1}{4\mu}\frac{dp}{dz}\left\{r^2 - r_1^2 + \frac{r_2^2 - r_1^2}{\log(r_2 / r_1)}\log\left(\frac{r}{r_2}\right)\right\}.$$

(30)