C.S.E-Mains 2000

MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each section.

PAPER–I

SECTION-A

1. Attempt any five of the following:

(a) Let V be a vector space over R an
Let $T=\{(x,y)\mid x,y\in V\}$
Define addition in T component wise and scalar multiplication by a complex number $\alpha+i\beta$, by $(\alpha+i\beta)(x,y)=(\alpha x-\beta y,\beta x+\alpha y)\forall\alpha,\beta\in R$ show that T is a vector space over C.

(b) Show that if λ is a characteristic root of a nonsingular matrix A, then λ^{-1} is a characteristic root of A^{-1}

(c) Use the Mean value theorem to prove that $\frac{2}{7}<\log 1.4<\frac{2}{7}$.

(d) Show that
$\iint x^{2l-1}y^{2m-1}dxdy=\frac{1}{4}r^{2(l+m)}\frac{\Gamma/\Gamma m}{\Gamma(l+m+1)}$ for all positive values of x and y lying inside the circle $x^2+y^2=r^2$

(e) Find the equations of the planes bisecting the angles between the planes $2x-y-2z-3=0$ and $3x+4y+1=0$ and specify the one which bisects the acute angle.

(f) Find the equation to the common conjugate diameters of the conics $x^2+4xy+6y^2=1$ and $2x^2+6xy+9y^2=1$.

2. (a) Prove that a real symmetric matrix A is positive definite if and only if $A=BB^t$ for some one-singular matrix B. Show also that

$$A=\begin{pmatrix}1&2&3\\2&5&7\\3&7&11\end{pmatrix}$$

Is positive definite and find the matrix B such that $A=BB^t$. Here B^t stands for the transpose of B.

(b) Prove that a system AX = B of a non homogeneous equations in a unknowns has a unique solution provided the coefficient matrix is not singular.

(c) Prove that two similar matrices have the same characteristic roots. Is its converse true? Justify your claim.

(d) Reduce the equation
$x^2+y^2+z^2-2xy-2yz+2zx+x-y-2z+6=0$ into canonical from and determine the nature of the quadric.

3. (a) Find the centre of gravity of the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. If the density varies as xyz.

(b) Let

$$f(x)=\begin{cases}0, & x \text{ is irrational}\\ 1; & x \text{ is rational}\end{cases}$$

Show that f is not Riemann-integrable on [a, b]

(c) Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(-1)^n\frac{n!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}\ldots\ldots\frac{1}{n}\right)$$

(d) Find constants a and b for which

$F(a,b)=\int_0^{\pi}\left\{\sin x-\left(ax^2+bx\right)\right\}^2 dx$ is a minimum.

4. (a) If the m^{th} derivative of r with respect to 's' is given by

$$r^{(m)}=a_m t+b_m n+c_m b,$$

Prove the reduction formulae:

$$a_{m+1}=a'_m-Kb_m$$

$$b_{m+1}=b'_m-Ka_m-\tau c_m$$

$$c_{m+1}=c'_m+\tau b_m$$

Where symbols have their usual meanings.

(b) Find the equation of the sphere through the circle

$x^2+y^2+z^2=4,\ x+2y-z=2$ and the point (1, -1, 1).

(c) A variable straight line always intersects the lines $x=c, y=0; y=c, z=0 z=c, x=0$.

Find the equation to its locus.

(d) Show that the locus of mid-points of chords of the cone

$$ax^2+by^2+cz^2=2fyz+2gzx+2hxy=0$$

Drawn parallel to the line $\frac{x}{1}=\frac{y}{m}=\frac{z}{n}$ is the plane

$$(al+hm+gn)x(hl+bm+fn)y+(gl+fm+cn)z=0$$

SECTION - B

5. Attempt any five of the following:

(a) Show the

$$3\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-8y=0$$

Has an integral which is a polynomial in x. Deduce the general solution.

(b) It T is the tension at any point P of a catenary and T_0 that at the lowest point c, then show that $T^2-T_0^2=W^2$, where W is the weight of the arc CP of the Catenary.

(c) Prove that a central force motion is a motion in a phane and the areal velocity of a particle is constant.

(d) A trapezoidal plate having its parallel sides of length x and y, (x, > y) at a distance z apart, is immersed vertically in water into x side uppermost (horizontal) at a depth d below the water surface. Find the total thrust on the surface.

(e) In what direction from the point (-1, 1, 1) is the directional derivative of $f=x^2yz^3$ a maximum? Compute its magnitude.

(f) (i) Show that the vovariant derivatives of the fundamental metric tensors.

g_{ij}, g^{ij}, g_i^j vanish.

(ii) Show that simultaneity is relative in special relativity theory.

6. (a) Reduce $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ where P, Q, R are functions of x, to the normal form. Hence solve

$$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+\left(4x^2-1\right)y=-3ex^2\sin 2x.$$

(b) Solve the differential equation $y = x - 2ap + ap^2$. Find the singular solution and interpret it geometrically.

(c) Show that $(4x+3y+1)dx+(3x+2y+1)dy=0$ represents a family of hyperbolas with a common axis and tangent at the vertex.

(d) Solve

$$x\frac{dy}{dx}-y=(x-1)\left(\frac{d^2y}{dx^2}-x+1\right)$$

by the method of variation of parameters.

7. (a) A telephonic wire weighing 0.04 lb per foot has a horizontal span of 150 feet and sag of 1.5 feet. Find the length of the wire and also find maximum tension.

(b) Assuming that the earth attracts points inside it with a force which varies as the distance from its centre, show that, if a straight frictionless airless tunnel be made from one point of the earth's surface to any point, a train would traverse the tunnel in slightly less than three quarter of an hour. Assume the earth to be a homogeneous sphere of radius 6400 km.

(c) A small bead is projected with any velocity along the smooth circular wire under the action of a force varying inversely as the fifth power of the distance from a centre of force situated on the circumference. Prove that the pressure on the wire is constant.

(d) A conical buoy 1 meter long, and of base diameter 1.2 meter, floats in water with its apex downwards. Determine the minimum weight of the buoy, for stable equilibrium.

8. (a) Show that

(i) $(A + B).(B + C) \times (C + A) = 2A.B \times C$

(ii) $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) - (A.\nabla)B + A(\nabla.B)$

(b) Evaluate $\iint_S F \cdot n\, ds$,

Where $F = 2xyi + yz^2 j + xzk$ and S is the surface of the parallelepiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1$ and $z = 3$.

(c) If g_{ij} and y_{ij} are two metric tensors defined at a point and $\left\{ {1 \atop ij} \right\}$ and $\left[{1 \atop ij} \right]$ are the corresponding Christoffel symbols of the second kind, then prove that $\left\{ {1 \atop ij} \right\} - \left[{1 \atop ij} \right]$ is a mixed tensor of the type A^1_{ij}.

(d) Establish the formula $E = mc^2$, the symbols have their usual meaning.

MATHEMATICS

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PAPER - II

SECTION A

1. Attempt any five of the following:

(a) Let n be a fixed positive integer and let Z_n be the ring of integers modulo n. Let $G = \{\bar{a} \in Z_n a \mid a \neq 0$ and a is relativity prime to n}. Show that G is a group under multiplication defined in Z_n. Hence, or otherwise, show that $a^{\phi(n)} \equiv a$ (mod n) for all integers a relatively prime to n where $\phi(n)$ denotes the number of positive integers that are less than n and are relatively prime to n.

(b) Let M be a subgroup and N a normal subgroup of a group G. Show that MN is a subgroup of G and MN/N is isomorphic to $M/(M \cap N)$.

(c) Given that the terms of a sequence $\{a_n\}$ are such that each a_k, $k \geq 3$, is the arithmetic mean of its two immediately preceding terms. Show that the sequence converges. Also find the limit of the sequence.

(d) Determine the vlues of x for which the infinite product

$$\prod_{n=0}^{\infty}\left(1+\frac{1}{x^{2^n}}\right)$$

converges absolutely. Find its value whenever it converges.

(e) Show that any for given points of the complex plane can be carried by a bilinear transformation to positions 1, -1, k and –k where the value of k depends on the given points.

(f) (i) Using Newton-Raphson's method, show that the iteration formula for finding the reciprocal of the p^{th} root of N is

$$x_{i+1} = \frac{x_i\left(p+1-Nx_i^p\right)}{p}$$

(ii) Prove De Morgan's Theorem

$$(p+q)' = p'.q'$$

by means of a truth table.

2. (a) Let F be a finite field. Show that the characteristic of F must be a prime integer p and the number of elements in F must be p^m for some positive integer m.

(b) Let F be a field and F[x] denote the set of all polynomials defined over F. If f(x) is an irreducible polynomial in F[x], show that the ideal generated by f(x) in F[x] is maximal and F[x]/(f(x)) is a field.

(c) Show that any finite commutative ring with no zero divisors must be a field.

3. (a) (i) Suppose f is twice differentiable real valued function in $(0, \infty)$ and M_0, M_1 and M_2 the least upper bounds of $|f(x)|, |f'(x)|$ and $|f''(x)|$ respectively in $(0, \infty)$. Prove for each $x > 0$, $h > 0$ that

$$f'(x) = \frac{1}{2h}\left[f(x+2h) - f(x)\right] - hf''(u)$$

For some $u \in (x, x+2h)$. Hence show that $M_1^2 \le 4M_0M_2$

(ii) Evaluate $\iint_S \left(x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z dx\, dy\right)$

By transforming into triple integral where S is the closed surface formed by the cylinder $x^2 + y^2 = a^2, 0 \le z \le b$ and the circular discs $x^2 + y^2 \le a^2, z = 0$ and $x^2 + y^2 \le a^2, z = b$.

(b) Suppose $f(\xi)$ is continuous on a circle C. Show that

$$\int_C \frac{f(s)\,ds}{(s-z)}$$

as z varies inside of C, is differentiable under the integral sign. Find the derivative. Hence of otherwise, derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside of C.

4. (a) (i) Evaluate $\int_0^1 \frac{dx}{1+x^2}$, by subdividing the interval (0, 1) into 6 equal parts and using Simpson's one-third rule. Hence find the value of π and actual error, correct to five places of decimals.

(ii) Solve the following system of linear equations, using Gauss-elimination method:

$$x_1 + 6x_2 + 3x_3 = 6$$
$$2x_1 + 3x_2 + 3x_3 = 117$$
$$4x_1 + x_2 + 2x_3 = 283.$$

(b) (i) Let

$$A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 7 & 4 \\ 1 & 4 & 9 \end{bmatrix}, B = A^1 \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Write a BASIC program that computes the inverse of A, determinant of A and the product of the matrix and its inverse.

(ii) Write a basic program to evaluate the formula

$$y = \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 \frac{1}{4}\left(\frac{x-1}{x}\right)^4 + \frac{1}{5}\left(\frac{x-1}{x}\right)^5$$

SECTION - B

5. Attempt any five of the following:

(a) Solve:

$$pq = x^m y^n z^{2l}$$

(b) Prove that if $x_1^3 + x_2^3 + x_3^3 = 1$ when $z = 0$, the solution of the equation

$$(S - x_1)p_1 + (S - x_2)p_2 + (S - x_3)p_3 = S - z$$

Can be given in the form

$$S^3\left\{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3\right\}^4 = (x_1 + x_2 + x_3 - 3z)^3$$

Where

$$S = x1 + x2 + x3 + z \text{ and } p1 = \frac{\partial z}{\partial x_1}, i = 1, 2, 3.$$

(c) Find the moment of inertia of an elliptic area about a line CP inclined at θ to the major axis and about a tangent parallel to CP, where C is the centre of the ellipse.

(d) Determine the stream lines and the path lines of the particle when the components of the velocity field are given by

$$u = \frac{x}{1+t}, v = \frac{y}{2+t}, \text{and } w = \frac{z}{3+t}.$$

Also state the condition for which the stream lines are identical with the path lines.

(e) (i) Let A and B be the two events such that

$$P(A) = \frac{3}{4} \text{ and } P(B) = \frac{5}{8}.$$

Show that

$$P(A \cup B) \geq \frac{3}{4}$$

And $= \frac{3}{8} \geq P(A \cap B) \geq \frac{5}{8}$

(ii) If X is a N(0, 1), find the probability density function of |X|.

(f) Two players each take out one or two matches and guess how many matches the opponent has taken. If one of the players guesses correctly, then the loser has to pay him as many rupees as the sum of the number held by both players; otherwise the payment is zero. Write down the pay-off matrix and obtain the optimal strategies of both players. Also find the value of the game.

6. (a) (i) Solve by Charpit's method the equation

$$p^2x(x-1) + 2pqxy + q^2y(y-1) - 2pxz - 2qyz + z^2 = 0.$$

(ii) Solve:

$$\left(D^2 - DD' - 2D'^2\right)z = 2x + 3y + e^{3x+4y}.$$

(b) A tightly stretched string with fixed end points x = 0, x = l is initially at rest in equilibrium position. If it is set vibrating by giving each point x of it a velocity kx(l - x), obtain at time t the displacement y at a distance x from the end x = 0.

7. (a) A plank of mass, m, is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end walks down the plank so that it does not move. Show that the gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

Where a is the length of the plank.

(b) Define irrotational and rotational flows giving an example for each.

Show that

$$u = \frac{-2xyz}{\left(x^2+y^2\right)^2}, v = \frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}, w = \frac{y}{\left(x^2+y^2\right)},$$

are the velocity components of a possible liquid motion.

Examine this for irrotational motion.

8. (a) An explosion in a factory manufacturing explosives can occur because of

(i) leakage of electricity,
(ii) defects in machinery,
(iii) carelessness of workers or
(iv) sabotage.

The probability that there is a leakage of electricity is 0.20, the machinery is defective is 0.30, the workers are careless is 0.40, there is sabotage is 0.10. The engineers feel that an explosion can occur with probability

(i) 0.25 because of leakage of electricity,

(ii) 0.20 because of defects in machinery,
(iii) 0.50 because of carelessness of workers and
(iv) 0.75 because of sabotage.

Determine the most likely cause of explosion.

(b) Two unbiased coins are tossed once (independently) and the number X of heads that turned up is noted. A number is selected at random from X, X + 1 and X + 2. If Y is the number selected, find the joint distribution of X and Y. Also obtain the expectation of XY.

(c) Solve the following assignment problem for the given assignment costs:

		Person				
		I	**II**	**III**	**IV**	**V**
	1	11	17	8	16	20
	2	9	7	12	6	15
Job	**3**	13	16	15	12	16
	4	21	24	17	28	26
	5	14	10	12	11	13