MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

*Candidates should attempt **FIVE** Questions,*
Selecting at least one from each of the Sections A, B and C.
All questions carry equal marks.

PAPER-I

SECTION-A

1. (a) Let V be the vector space of functions from R to R(real number). Show that f, g, h in V are linearly independent where

$$f(t) = e^{2t},\ g(t) = t^2 \text{ and } h(t) = t.$$

(b) If the matrix of a linear transformation T on $V_2(R)$ with respect to the basis $B = \{(1,0),(0,1)\}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then what is the matrix of T with respect to the ordered basis $B_1 = \{(1,1),(1,-1)\}$?

(c) Diagonalize the matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}.$$

2. (a) Test for congruency of the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Prove that $A^{2n} = B^2m = 1$ when m and n are positive integers.

(b) If A is a skew symmetric matrix of order n, prove that $(I - A)(I + A)^{-1}$ is orthogonal.

(c) Test for the positive definiteness of the quadratic form

$$2x^2 + y^2 + 2z^2 + 2xy - 2zx.$$

3. (a) Determine the set of all points where the function

$$f(x) = \frac{x}{1+|x|} \text{ is differentiable.}$$

(b) Find three asymptotes of the curve

$$x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y - 10 = 0.$$

Also find the intercept of one asymptote between the other two.

(c) Find the dimensions of a right circular cone of minimum volume which can be circumscribed about a sphere of radius a.

4. (a) If f is Riemann integral over every interval of finite length and f(x + y) = f(x) + f(y) for every pair of real numbers x and y, show that

$$f(x) = c\,x \text{ where } c = f(1).$$

(b) Show that the area bounded by cissoid

$$x = a\sin^2 t,\ y = a\frac{\sin^3 t}{\cos t} \text{ and its asymptote is } \frac{3\pi a^2}{4}$$

(c) Show that $\iint x^{m-1}y^{n-1}dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{a^n b^n}{4}\frac{\Gamma(m/2)\Gamma(n/2)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+2\right)}.$$

SECTION – B

5. (a) If P and D are the ends of a pair of semi-conjugate diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

show that the tangents at P and D meet on the ellipse $x^2 / a^2 + y^2 / b^2 = 2$.

(b) Find the equation of the cylinder whose generators touch the sphere $x^2 + y^2 + z^2 = 9$ and are perpendicular to the plane $x - y - 3z = 5$.

(c) Calculate the curvature and torsion at the point u of the curve given by the parametric equations

$x = a(3u-u^3)$ $y = 3au^2$

$z = a(3u + u^2)$.

6. (a) Solve the differential equation

$$\frac{xdx+ydy}{xdy-ydy}=\left(\frac{1-x^2-y^2}{x^2+y^2}\right)^{1/2}$$

(b) Solve $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x$.

(c) By the method of variation of parameters solve the differential equation

$$\frac{d^2y}{dx^2}+a^2y=\sec(ax)$$

7. (a) If $\overline{a},\overline{b},\overline{c}$ are the position vectors of A, B, C prove that

$$\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}$$

is a vector perpendicular to the plane ABC.

(b) Evaluate $\overline{F}=\nabla\left(x^3+y^3+z^3-3xyz\right)$, find $\nabla\times\overline{F}$

(c) Evaluate $\int_C\left(e^{-x}\sin y\,dx+e^{-x}\cos y\,dy\right)$

(by Green's theorem), where C is the rectangle whose vertices are

$(0, 0)$, $(\pi, 0)$, $(\pi, \pi/2)$ and $(0, \pi./2)$.

8. (a) If X, Y, Z are the components of a contravariant vector in rectangular Cartesian coordinates x, y, z in a three dimensional space, show that the components of the vector in cylindrical coordinates r, θ, z are

$$X\cos\theta+Y\sin\theta,\frac{-X}{r}\sin\theta+\frac{Y}{r}\cos\theta,Z.$$

(b) Show that the inner product of two tensors A_i^p and B_t^{qs} is a tensor of rank three.

(c) Show that $\nabla^2\phi=g^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j}-\frac{\partial\phi}{\partial x^l}\left\{\begin{matrix}l\\ij\end{matrix}\right\}\right)$

where ϕ is a scalar function of coordinates x.

SECTION - C

9. (a) A perfectly rough plane is inclined at an angle α to the horizon. Show that the least eccentricity of the ellipse which can rest on the plane is

$$\left[\frac{2\sin\alpha}{1+\sin\alpha}\right]^{1/2}.$$

(b) A string of length a forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is

$$\frac{2W\left(2b^2-a^2\right)}{b\left(4b^2-a^2\right)^{1/2}}.$$

(c) A uniform chain, of length ℓ and weight W, hangs between two fixed points at the same level, and weight W is attached at the middle point. If K be the sag in the middle, prove that the pull on either point of support is

$$\frac{K}{2l}W+\frac{\ell}{4K}W'+\frac{\ell}{8K}W.$$

10. (a) If in a simple harmonic motion u, v, w be the velocities at a distance a, b, c from a fixed point on the straight line (which is not the centre of force), show that the period T is given by the equation

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

(b) A particle moves with a central acceleration $\mu(r^5 - c^4 r)$, being projected from an apse at distance c with a velocity $\sqrt{2\mu/3}c^3$. Determine its path.

(c) A particle of mass m projected vertically under gravity, the resistance of air being mk (velocity). Show that the greatest height attained by the particle is $V^2/g\,[\lambda - \log(1+\lambda)]$ where V is the terminal velocity of the particle and λV is the initial vertical velocity.

11. (a) An ellipse is just immersed in water (touching water surface) with its major axis vertical. Show that if the centre of pressure coincides with the focus the eccentricity of ellipse is ¼.

(b) Two solids are each weighed in succession in three homogeneous liquids of different densities. If the weights of the one are w_1, w_2, w_3 and those of the other are W_1, W_2 and W_3, prove that

$$w_1(W_2 - W_3) + w_2(W_3 - W_1) + w_3(W_1 - W_2) = 0.$$

(c) Masses m and m′ of two gases, in which the ratio of the pressure to the density (p/ρ) are respectively k and k′, are mixed at the same temperature. Prove that the ratio of the pressure to the density in the compounds is $\frac{mk++m'k'}{m+m'}$.

12. (a) Derive the Lorentz transformation equations.

(b) If u and v are two velocities in the same direction and V is their resultant velocity given by

$$\tan\mathrm{h}^{-1}V/c=\tan\mathrm{h}^{-1}\frac{u}{c}+\tan h^{-1}\frac{v}{c},$$

then deduce law of composition of velocities from this equation.

(c) (i) Define relativistic energy and momentum and establish $E^2 = p^2c^2 + m_0^2 c^4$ with usual notation.

(ii) Two lumps of clay each of rest mass m_0 collide head- on with velocity 3/5 c, and stick together. What is the mass of the composite lump?

MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt any five Questions.
ALL Questions carry equal marks.

PAPER - I

SECTION - A

1. (a) If ϕ is a homomorphism of G into $\overline{G}$ with kernel K, then show that K is a normal subgroup of G.
   (b) If p is a prime number and $p^{\alpha} | o(G)$, then prove that G has a subgroup of order p^{α}.
   (c) Let R be a commutative ring with unit element whose only ideals are (o) and R itself. Show that R is a field.
2. (a) Let A be a subset of the metric space (M, ρ). If (A, ρ) is compact, then show that A is a closed subset of (M, ρ)
   (b) A sequence $\{S_n\}$ is defined by the recursion formula $S_{n+1} = \sqrt{3S_n}, S_1 = 1$. Does this sequence converge? If so, find lim S_n.
   (c) Test for convergence the integral
   $$\int_0^1 x^p (\log 1/x)^q \, dx.$$
3. (a) Find the shortest distance from the origin to the hyperbola
   $x^2 + 8xy + 7y^2 = 225$, $z = 0$.
   (b) Show that the double integral $\iint_R \frac{x-y}{(x+y)^3} dxdy$ does not exist over R = [0, 1; 0, 1]
   (c) Verify the Gauss divergence theorem for
   $F = 4x\, e_x - 2y^2\, e_y + z^2\, e_z$
   taken over the region bounded by
   $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.
   where, e_x, e_y, e_z are unit vectors along x-, y- and z-directions respectively.
4. (a) Examine the nature of the function
   $$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}, z \neq 0$$
   $f(0) = 0$
   in a region including the origin and hence show that Cauchy-Riemann equations are satisfied at the origin but f(z) is not analytic there.
   (b) For the function
   $$f(z) = \frac{-1}{z^2 - 3z + 2}$$
   find the Laurent series for the domain
   (i) $1 < |z| < 2$,
   (ii) $|z| > 2$.

Show further that

$$\oint_c f(z)dz = 0$$

where C is any closed contour enclosing the points z = and z = 2.

(c) Show that the transformation

$$w = \frac{2z+3}{z-4}$$

transforms the circle $x^2 + y^2 — 4x = 0$ into the straight line 4u+3 =0, where w = u+ iv.

5. (a) Using Residue theorem show that

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4} dx = \frac{\pi}{2} e^{-a} \sin a, (a > 0)$$

(b) The function f(z) has a double pole at z = 0 with. residue 2, a simple pole at z = 1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If f(2) = 5 and f(-1) = 2, find f(z).

(c) What kind of singularities the following functions have?

(i) $\frac{1}{1-e^z}$ at $z = 2\pi i$

(ii) $\frac{1}{\sin z - \cos z}$ at $z = \pi / 4$

(iii) $\frac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z=\infty$

In case (iii) above what happens when a is an integer (including a = 0) ?

SECTION - B

6. (a) Verify that the differential equation

$(y^2 + yz)dx + (xz + z^2)dy + (y^2 — xy)dz = 0$

is integrable and find its primitive.

(b) Find the surface which intersects the surfaces of the system

z(x + y) = c(3z + 1), c = a constant,

orthogonally and which passes through the circle $x^2 + y^2 = 1$, z = 1.

(c) Find the characteristics of the equation pq = z, and determine the integral surface which passes through the parabola x= 0, $y^2 = z$.

7. (a) Use Charpit's method to find a complete integral of

$p^2 + q^2 — 2px —2qy + 1 = 0$.

(b) Find the solution of the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = e^{-x} \cos y$$

which $\to 0$ as $x \to \infty$ an has the value cos y when x = 0.

(c) One end of a string (x = 0) is fixed, and the point x = a is made to oscillate, so that at time t the displacement is g(t). Show that the displacement u(x, t) of the point x at time t is given by

u(x, t) = f(ct — x) — f(ct + x).

where f is a function satisfying the relation

$$f(t+2a) = f(t) - g\left(\frac{t+a}{c}\right).$$

8. (a) A particle of given mass m moves in space with the Lagrangian

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V + \dot{x}A + \dot{y}B + \dot{z}C$$

where V, A, B, C are given functions of x, y, z. Show that the equations of motion are

$$m\ddot{x} = -\frac{\partial V}{\partial x} + \dot{y}\left[\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right] - \dot{z}\left[\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right]$$

and two similar equations for y and z. Find also the Hamiltonian H in terms of generalized momenta.

(b) A wheel consists of a thin rim of mass M and n evenly placed spokes each of mass m, which may be considered as thin rods terminating at the centre of the wheel. If the wheel is rolling with linear velocity v, express its kinetic energy in terms of M, m, n, v. With what acceleration will it roll down a rough inclined plane of inclination α ?

(c) Find the moment of inertia of a solid hemisphere about a diameter of its plane base. A solid hemisphere is held with its base against a smooth vertical wall and its lowest point on a smooth floor. The hemisphere is released. Find the initial reactions on the wall and the floor.

9. (a) Derive the equation of continuity for a fluid in which there are no sources or sinks, liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + kx^{2s}/a$ about the x-axis ($-a \le x \le a$). If the liquid enters at the end $x = -a$ of the pipe with velocity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from x=-a to x = +a is

$$\left\{\frac{2a}{V(1+k)^2}\right\}\left(1+\frac{2}{3}k+\frac{1}{5k^2}\right)$$

(Assume that k is so small that the flow remains appreciably one-dimensional throughout).

(b) A spherical globule of gas initially of radius R_o and at pressure P_o expands in an infinite mass of water of density ρ in which the pressure at infinity is zero. The gas is initially at rest and its pressure p and volume v are governed by the equation $pv^{4/3}$ = constant. Prove that the gas doubles its radius in time $\frac{28R_o}{15}\left(\frac{2\rho}{P_o}\right)^{1/2}$.

(c) Two sources each of strength m are placed at points (-a, 0) and (a, 0) and a sink of strength 2m is placed at the origin. Show that the stream lines are the curves

$$(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$$

Where λ is a variable parameter.

10. (a) Obtain the Simpson's rule for the integral

$$I = \int_a^b f(x)dx$$

and show that this rule is exact for polynomials of degree $n \le 3$. In general show) that the error of approximation for Simpson's rule is given by

$$R = -\frac{(b-a)^5}{2880} f^{iv}(\eta) \qquad \eta \in (0, 2)$$

Apply this rule to the integral $\int_0^1 \frac{dx}{1+x}$ and show that $|R| \le 0.008333$.

(b) Using fourth order classical Runge-Kutta method for the initial value problem.

$du / dt = - 2t\, u^2$, $u(0) = 1$,

with h = 0.2 on the interval [0, 1], calculate u(0.4) correct to six places of decimal.

(c) In an examination students are awarded letter grades according to the following scheme:

Score Range	**Letter Grade**
Score $\geq$ 90	A
80 $\geq$ Score < 90	B
70 $\geq$ Score < 80	C
60 $\geq$ Score < 70	D
Score > 60	F

For each score print a message identifying the letter grade. For grade F an additional message "This is a failing case" may be mentioned. Prepare the flowchart for this scheme and write the program in BASIC.

SECTION - C

11. (a) Suppose that the universal set U is given by U = {x| 0 ≤ x ≤ 2}. Let the sets A and B be defined as follows:

A={x | 1/2 < x≤1} and B = {x|1/4 ≤ x < 3/2}.

Describe the following sets:

(i) $\overline{A \cup B}$

(ii) $A \cup \overline{B}$

(iii) $\overline{A \cap B}$

(iv) $\overline{A} \cap B$

(b) Suppose that X has pdf

f(x) = 2x, 0 < x < 1

= 0, elsewhere.

Let H(x) = 3X + 1. Obtain the pdf of Y = H(X).

(c) Suppose that the two-dimensional continuous random variable (X, Y) has joint pdf given by

$$f(x,y) = x^2 + \frac{xy}{3}, \qquad 0 \leq x \leq 1, 0 \leq y \leq 2$$
$$= 0 \qquad \text{elsewhere}$$

Find P(B) where B = {X + Y ≥ 1}.

12. (a) Eight coins were tossed together, and the number x of heads resulting was observed. The operation was performed 256 times: and the frequencies that were obtained for the different values of x are shown in the following table. Calculate measures of central tendency and mean deviation about mean :-

x	**f**
1	9
2	26
3	59
4	72
5	52

6	29
7	7
8	1
	256

(b) A printing machine can print n "letters", say $\alpha_1, \alpha_2, \ldots, \alpha_n$. It is operated by electrical impulses, each letter being produced by a different impulse. Assume that there exists a constant probability p of printing the correct letter and also assume independence. One of the n impulses, chosen at random, was fed into the machine twice and both times the letter α_1 was printed. Compute the probability that the impulse chosen was meant to print α_1.

(c) If two independent variates, X and Y, have Poisson distributions with means m_1 and m_2, find the distribution of the sum X + Y.

13. (a) A certain cubical die was thrown 9000 times, and a 5 or a 6 was obtained 3,240 times. Find the standard, error of the proportion of successes in 9000 throws; and deduce that the probability of throwing a 5 or a 6 in a single trial almost certainly lies between 0.345 and 0.375. Can we say the die is unbiased?

(b) Suppose that X has distribution N(3, 4). Find a number C such that

$$P(X > C) = 2\,P(X \le C).$$

(c) A physical quantity is measured many times for accuracy. Each measurement is subject to a random error. It is judged reasonable to assume that it is uniformly distributed between -1 and + 1 in a conveniently chosen unit. How many measurements should be taken in order that the probability will exceed 0.95 that the average will differ from the true value by at most 0.2?

14. (a) A police department has the following minimal daily requirements for police officers during its six shift periods:

Time of Day	**Period**	**Minimal Number Required**
2 a.m. - 6 a.m.	1	22
6 a.m. - 10 a.m.	2	55
10 a.m. - 2 p.m.	3	88
2 p.m. - 6 p.m.	4	110
6 p.m. - 10 p.m.	5	44
10 p.m. - 2 a.m.	6	33

An officer must start at the beginning of a 4-hour shift and stay on duty for two consecutive shifts (an 8-hour tour). Any one starting during period 6 stays on duty during period 1 of the next day. The objective of the police department is to always have on duty the minimal number required in a period but to do so with the least number of officers. Develop the corresponding linear programming model.

(b) Show that a problem in the theory of games can be expressed as a linear programming problem.

(c) Respond True or False to the following, justify your answer in case of False:

(i) If the number of primal variables is much smaller than the number of constraints, it is more efficient to obtain the solution of the primal by solving its dual.

(ii) When the primal problem is non-optimal, the dual problem is automatically infeasible.

(iii) An unrestricted primal variable will have the effect of yielding an equality dual constraints.

(iv) If the solution space is unbounded, the objective value always will be unbounded.

(v) The selection of the entering variable from among the current non-basic variables as the one with the most negative objective coefficient guarantees the most increase in the objective value in the next iteration.

(vi) In the simplex method, the feasibility condition for the maximization and minimization problems are different.

(vii) A simplex iteration (basic solution) may not necessarily coincide with a feasible extreme point of the solution space.

(viii) If the leaving variable does not correspond to the minimum ratio, at least one basic variable will definitely become negative in the next iteration.

15. (a) Develop mathematical model of a balanced transportation problem. Prove that it always has a feasible solution.

(b) Find the optimal assignment for the given assignment costs:

Machine

Job	1	2	3
1	5	7	9
2	14	10	12
3	15	13	16

(c) Give the economic interpretation of duality in linear programming.

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