MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt any FIVE questions.
All questions carry equal marks.

1. (a) R^4, let W_1 be the space generated by (1, 1, 0,–1), (2, 6,0) and (-2, -3, -3, 1) and let W_2 be the space generated by(-1,-2,–2,2), (4;6,4,-6) and (1,3,4,-3). Find a basis for the space $W_1 + W_2$.

   (b) Let V be a finite dimensional vector space and $v \in V$, $v \neq 0$. Show that there exists a linear functional f on V such that $f(v) \neq 0$.

   (c) Let $V = R^3$ and v_1, v_2, v_3 be a basis of R^3. Let $T : V \to V$ be a linear transformation such that $T(v_1)=v_1+v_2+v_3$, $1 \leq i \leq 3$. By writing the matrix of T with respect to another basis, show that the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ is similar to } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. (a) Let $V= R^3$ and $T: V \to V$ be the linear map defined by $T(x, y, z) = (x + z, -2x + y, -x +2y+z)$. What is the matrix of T with respect to the basis (1, 0, 1), (-1, 1, 1) and (0, 1, 1)? Using this matrix, write down the matrix of T with respect to the basis (0, 1, 2), (-1, 1, 1) and (0, 1, 1).

   (b) Let V and W be finite dimensional vector spaces such that dim V ≥ dim W. Show that there is always a linear map of V onto. W.

   (c) Solve

$$x+y-2z=1$$
$$2x-7z=3$$
$$x+ y-z=5$$

   by using Cramer's rule.

3. (a) Find the inverse of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ by computing}$$

   its characteristic polynomial.

   (b) Let A and B be $n \times n$ matrices such that AB = BA. Show that A and B have a common characteristic vector.

   (c) Reduce to canonical form the orthogonal matrix

$$\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

4. (a) Find the asymptotes of the curve
$4(x^4+ y^4) - 17x^2y^2 – 4x(4y^2 – x^2) + 2(x^2 – 2) = 0$ and show that they pass through the points of intersection of the curve with the ellipse $x^2 + 4y^2 = 4$.

(b) Show that any continuous function defined for all real x and satisfying the equation f (x) = f (2x + 1) for all x must be a constant function.

(c) Show that the maximum and minimum of the radii vectors of the sections of the surface

$$\left(x^2+y^2+z^2\right)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

by the plane

$$\lambda x + \mu y + vz = 0$$

are given by the equation

$$\frac{a^2\lambda^2}{1-a^2r^2}+\frac{b^2\mu^2}{1-b^2r^2}+\frac{c^2v^2}{1-c^2r^2}=0$$

5. (a) If $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

(b) Evaluate

$$\int_0^\infty\int_0^\infty \frac{e^{-y}}{y}\,dxdy$$

(c) The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid so formed.

6. (a) Find the equation of the common tangent to the parabolas $y^2=4ax$ and $x^2 = 4by$.

(b) If the normal at any point 't_1' of a. rectangular hyperbola $xy = c^2$ meets the curve again at the point 't_2', prove that $t_1^3t_2 = -1$.

(c) A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Through A, B, C the planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

7. (a) Find the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and has the smallest possible radius.

(b) The generators through a point P on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meet the principal elliptic section in two points such that the eccentric angle of one is double that of the other. Show that P lies on the curve

$$x=\frac{a\left(1-3t^2\right)}{1+t^2},y=\frac{bt\left(3-t^2\right)}{1+t^2},z=ct.$$

(c) A curve is drawn on a right circular cone, semi-vertical angle α, so as to cut all the generators at the same angle β. Show that its projection on a plane at right angles to the axis is an equiangular spiral. Find expressions for its curvature and torsion.

8. (a) Find the curves for which the sum of the reciprocals of the radius vector and polar sub-tangent is constant.

(b) Solve:

$$x^2\left(y-px\right)=yp^2, p\equiv\frac{dy}{dx}$$

(c) $y\sin 2x\,dx-\left(1+y^2+\cos^2 x\right)dy=0$

9. (a) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+10y+37\sin 3x=0$

Find the value of y when x = π/2, if it is given that y = 3 and $\frac{dy}{dx}=0$, when x =0.

(b) Solve:

$$\frac{d^4y}{dx^4}+2\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}=x^2+3e^{2x}+4\sin x$$

(c) Solve:

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x$$

10. (a) State and prove 'Quotient law' of tensors.

(b) If $x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$, show that.

(i) $\vec{r}\times grad f(r)=0$

(ii) $div(r^n\vec{r})=(n+3)r^n$

(c) Verify Gauss' divergence theorem for

$$\vec{F}=xy\hat{i}+z^2\hat{j}+2yz\hat{k}$$

on the tetrahedron

x=y=z=0, x+y+z=1

11. (a) A body of weight W is placed on a rough inclined plane whose inclination to the horizon is α greater than the angle of friction λ. The body is supported by a force acting in a vertical plane through the line of greatest slope and makes an angle θ with the inclined plane. Find the limits between which the force must lie.

(b) A body consisting of a cone and a hemisphere on the same base rests on a rough horizontal table, the hemisphere being in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the sphere.

(c) A hollow cone without weight, closed and filled with a liquid, is suspended from a point in the rim of its base. If ϕ be the angle which the direction of the resultant pressure makes with the vertical, then show that

$$\cot\phi=\frac{28\cot\alpha+\cot^3\alpha}{48}$$

α being the semi-vertical angle of the cone.

12. (a) One end of a light elastic string of natural length a and modulus 2 mg is attached to a fixed point 0 and the other to a particle of mass m. The particle is allowed to fall from the position of rest at 0. Find the greatest extension of the string and show that the particle will reach 0 again after a time

$$\left(\pi+2-\tan^{-1}2\right)\sqrt{\frac{2a}{g}}$$

(b) A stone is thrown at an angle α with the horizon from a point in an inclined plane whose inclination to the horizon is β, the trajectory lying in the vertical plane containing the line of greatest slope. Show that if θ be the elevation of that point bf the path which is most distant from the inclined plane, then

2 tan θ = tan α + tan β

(c) A particle moves under gravity on a vertical circle, sliding down the convex side of smooth circular arc. If its initial velocity is that due to a fall to the starting point from a height h above the centre; show that it will fly off the circle when at a height $\frac{2h}{3}$ above the centre.

MATHEMATICS

Time Allowed: 3 hours **Maximum Marks: 300**

Candidates should attempt any five Questions.
ALL Questions carry equal marks.

PAPER-I

SECTION - A

1. (a) Let R be the set of real numbers and
$G=\{(a,b) \mid a,b \in R, a \neq o\}$.
$* \; G \times G \to G$ is defined
by $(a, b) * (c, d) = (ac, be + d)$.
Show that (G, *) is a group. Is it abelian ?
Is (H, *) a subgroup of (G, *), when $H = \{(1, b) \mid b \in R\}$?

(b) Let f be a homomorphism of a group G onto a group G' with kernel H. For each subgroup K' of G' define K by
$K=\{x \in G \mid f(x) \in K'\}$.
Prove that
(i) K' is isomorphic to K / H.
(ii) G/ K is isomorphic to G' / K'.

(c) Prove that a normal subgroup H of a group G is maximal, if and only if the quotient group G/H is simple.

2. (a) In a ring R, prove that cancellation laws hold, if and only if R has no zero divisors.

(b) If S is an ideal of a ring R and T any subring or R, then prove that S is an ideal of $S+T=\{s+t \mid s \in S, t \in T\}$.

(c) Prove that the polynomial $x^2 + x + 4$ is irreducible over the field of integers modulo 11.

3. (a) Let F be the set of all real valued bounded continuous functions defined on the closed interval [0, 1]. Let d be a mapping of $F \times F$ into R, the set of real numbers, defined by

$$d(f,g)=\int_0^1 | f(x)-g(x) | \, dx$$

for all f, g in F. Verify that d is a metric for F.

(b) Prove that a compact set in a metric space is a closed set.

(c) Let C[a, b] denote the set of all functions f on [a, b] which have continuous derivatives at all points of I=[a, b]. For f, g, $\in$ C[a, b] define
$d(f,g) = f(a)-g(b)| + \sup\{|f'(x)-g'(x)|, x \in I\}$.
Show that the space (C[a, b], d) is complete.

4. (a) A function f is defined in the interval (a, b) as follows:
$f(x) = q^{-2}$, when $x=pq^{-1}$,
$f(x) = q^{-3}$, when $x=(pq^{-1})^{1/2}$
where p, q are relatively prime integers; f(x) = 0, for all other values of x.
Is f Riemann integrable ? Justify your answer.

(b) Test for uniform convergence, the series

$$\sum_{n=1}^{\infty} \frac{2^n x^{\left(2^n-1\right)}}{1+x^{2n}}$$

(c) Evaluate

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\,dx\,dy$$

5. (a) Sketch the ellipse C described in the complex plane by

$Z = A\cos\lambda t + iB\sin\lambda t$, $A>B$,

where t is a real variable and A, B, λ are positive constants. If C is the trajectory of a particle with z(t) as the position vector of the particle at time t, identify with justification

(i) the two positions where the acceleration is maximum, and

(ii) the two positions where the velocity is minimum.

(b) Evaluate

$$\lim_{z\to 0}\frac{1-\cos z}{\sin\left(z^2\right)}$$

(c) Show that z =0 is not a branch point for the function $f(z)=\frac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity?

6. (a) Prove that every polynomial equation

$a_0+a_1z+a_2z^2+\ldots+a_nz^n=0$, $a_n\neq 0$, $n\geq 1$ has exactly n roots.

(b) By using residue theorem, evaluate

$$\int_0^{\infty}\frac{\log_e\left(x^2+1\right)}{x^2+1}dx$$

(c) About the singularity z = -2, find the laurent expansion of

$$(z-3)\sin\frac{1}{z+2}$$

Specify the region of convergence and the nature of singularity at z=-2.

7. (a) (i) Find the differential equation of all spheres of radius λ having their centre in xy-plane.

(ii) Form differential equation by eliminating f and g from $z=f(x^2-y)+g(x^2+y)$.

(b) Solve: $z^2(p^2+q^2+1) = C^2$

(c) Find the integral surface of the equation

$$(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$$

passing through the curve $xz = a^3$, $y = 0$

8. (a) Apply Charpit's method to find the complete integral of

$z = px + ay + p^2 + q^2$.

(b) Solve:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\cos mx\cos ny\,.$$

(c) Find a surface passing through the lines z = x = 0 and z - 1 = x - y = 0 satisfying

$$\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2}=0\,.$$

SECTION – B

9. (a) A uniform rod OA of length 2a free to turn about its end 0 revolves with uniform angular velocity ω about the vertical OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or $\cos^{-1}\left(\dfrac{3g}{4a\omega^2}\right)$.

(b) Six equal uniform rods form a regular hexagon loosely jointed at the angular points, and rests on a smooth table, a blow is given perpendicular to one of them at its middle point. Show that the opposite rod begins to move with one- tenth of the velocity of the rod that is struck.

(c) A cylinder of mass m, radius R and moment of inertia 1 about its geometrical axis rolls down a hill without slipping under the action of gravity. If the velocity of the centre of mass of the cylinder is initially v_0, fine the velocity after the cylinder has dropped through a vertical distance h.

10. (a) A perfectly rough circular hoop of diameter 24 cm rolls on a horizontal floor with velocity V cm/sec towards an inelastic step of height 4 cm, the plane of the hoop being vertical and perpendicular to the edge of the step. Prove that hoop can mount the step without losing contact at any stage if $2.4\sqrt{2g} > V > 2.4\sqrt{g}$.

(b) A homogeneous sphere rolls down an imperfectly rough fixed sphere, starting from rest at the highest point. If the sphere separates when the line joining their centres makes an angle 30^o with the vertical, show that the coefficient of friction μ satisfies the following equation:

$$e^{\mu\pi/3} = \frac{3\sqrt{3}+6\mu}{4\left(1-2\mu^2\right)}.$$

11. (a) Show that the motion specified by $\vec{q} = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$ is a possible form for an incompressible fluid. Determine the stream lines. Show that the motion is irrotational and find the velocity potential.

(b) A sphere is at rest in an infinite mass of homogeneous liquid of density ρ. the pressure at infinity being ϖ. Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is

(c) Find the stream function of two-dimensional motion due to two equal sources and an equal sink situated midway between them.

In a region bounded by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the end of one of the bounding radii. Show that the stream line leaving either end at an angle $\pi/6$ with the radius is

$$r^2 \sin\left(\frac{\pi}{6}+\theta\right) = a^2 \sin\left(\frac{\pi}{6}-\theta\right),$$

where a is the radius of the quadrant.

12. (a) Describe Newton-Raphson method for finding the solutions of the equation f(x) = 0 and show that the method has a quadratic convergence.

(b) The following are the measurements t made on a curve recorded by the oscillograph representing a change of current i due to a change in the conditions of an electric current:

t :	1.2	2.0	2.5.	3.0

i : 1.36 0.58 0.34 0.20

Applying an appropriate formula interpolate for the value of i when t = 1.6.

(c) Solve the system of differential equations

$$\frac{dy}{dx} = xz + 1, \qquad \frac{dz}{dx} = -xy$$

for x = 0.3 given that y= 0 and z =1 when x= 0, using Runge-Kutta method of order four.

13. (a) For a statistical distribution the mean, variance, γ_1 and β_2 are respectively 10, 16, +1 and 4. Find the first four moments about the origin.

(b) X_1 and X_2 are random variables with means μ_1, μ_2 equal variances σ^2 and positive correlation r. If for some given θ

$$U_1 = X_1 \cos\theta + X_2 \sin\theta$$

$$U_2 = -X_1 \sin\theta + X_2 \cos\theta$$

find the coefficient of correlation ρ between U_1 and U_2 and prove that $0 \le \rho \ge r$. Deduce that $\frac{X_1 + X_2}{\sqrt{2}}$ and $\frac{X_2 - X_1}{\sqrt{2}}$ are not correlated random variables even if X_1 and X_2 are correlated.

(c) For the data given below fit a straight line and a parabola of the second order using the principle of least squares and determine which of these curves is a better fit.

x :	0	1	2	3	4
y :	1	5	10	22	38

14. (a) If x be one of the first hundred natural numbers chosen at random, find the probability that

$$x + \frac{100}{x} > 50.$$

(b) Among the three hundred employees of a company 240 are union members while the others are not. If eight of the employees are to be chosen to serve on a committee which administers the pension fund, find the probability that five of them will be union members while the others are not, using

(i) hyper-geometric distribution and

(ii) binomial approximation.

(c) Genetic theory states that children having one parent of blood type M and the other of blood type N will always be of the three types M, MN or N and that the proportions of these three types will be on the average as 1 : 2 : 1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M, 45% of type MN and the remaining of type N. Test the validity of the genetic theory using chi-square test at 5% level of significance. The following extract from chi-square tables may be used.

Number of Degree of freedom	**Probability of a deviation greater than chi-square**							
	.01	.02	.05	.10	.90	.95	.98	.99
1	6.635	5.412	3.841	2.706	.0158	.00393	.000628	.000157
2	9.210	7.824	5.991	4.605	.211	.103	.0404	.0201
3	11.341	9.837	7.815	6.251	.584	.352	.185	.115
4	13.277	11.668	9.488	7.779	1.064	.711	.429	.297

15. (a) Solve that linear programming problem:

$$\text{Maximize} \quad z = 3x_1 + 5x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 \quad\quad &\leq 4 \\ x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1,\ x_2 &\leq 0.. \end{aligned}$$

If the cost coefficient of x_1 is kept fixed, find the range for the cost coefficient of x_2 without affecting the optimal solution.

A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth etc. The number of arrivals averages 80 persons in an eight hour service day. Each tax adviser spends an irregular amount of time serving the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes. Calculate the average number of people waiting to be serviced, average time a person spends in the system and the average waiting time for a person. What is the expected number of idle tax advisers at any specified time:

(c) Solve the assignment problem represented by the following for minimisation of costs. Find also alternate solutions if any.

	I	**II**	**III**	**IV**	**V**	**VI**
A	11	24	60	13	21	29
B	45	80	74	52	65	50
C	43	30	93	39	47	35
D	76	44	29	51	41	34
E	38	13	59	24	27	27
F	5	58	55	33	19	30

16. (a) A company has four plants P_1, P_2, P_3, P_4 from which it supplies to three markets M_1, M_2, M_3. Determine the optimal transportation plan using Modi method from the following data giving the plant to market shifting costs, quantities available at each plant and quantities required at each market:

		Plant				**Required at**
		P1	P2	P3	P4	Market
Market	M_1	21	16	25	13	11
	M_2	17	18	14	23	13
	M_3	32	27	18	41	19
Available at plant		6	10	12	15	43

(b) Determine the maximum value of

$$z=P_1P_2 \ldots P_n$$

subject to the constraints

$$\sum_{i=1}^{n} c_i p_i \leq x,\ \ 0 \leq p_i \leq 1(i=1,2,....,n)$$

(assume that $c_i > x$ for all i)

(c) Determine the optimal sequence of jobs that minimizes the total elapsed time required to complete the following jobs and find the total elapsed time. The jobs are to be processed on three machines M_1, M_2, M_3 in the same order M_1, M_2, M_3 and processing times are as below:

		Job						
		A	**B**	**C**	**D**	**E**	**F**	**G**
Machine	**M_1**	3	8	7	4	9	8	7
Machine	**M_2**	4	3	2	5	1	4	3
Machine	**M_3**	6	7	5	11	5	6	12

Find also the idle times for the -three machines.

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