MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt any FIVE questions.
All questions carry equal marks.

PAPER-I

1. (a) Show that $f_1(t) = 1$, $f_2(t) = t-2$, $f_3(t) = (t-2)^2$ form a basis of P_3, the space of polynomials with degree ≤ 2. Express $3t^2-5t+4$ as a linear combination of f_1, f_2, f_3.

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(b) If $T : V_4(R) \rightarrow V_3(R)$ is a linear transformation defined by
$T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$
For a, b, $d \in R$, then verify that
Rank T + Nullity T = dim $V_4(R)$.

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(c) If T is an operator on R_3 whose basis is
$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ such that

$$[T:B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

find the matrix of T with respect to a basis
$B_1 = \{(0,1,-1), (1,-1,1), (-1,1,0)\}$.

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2. (a) If $A = [a_{ij}]$ is an n x n matrix such that
$a_{ii} = n$, $a_{ij} = r$ if $i \neq j$, show that

$$[A-(n-r)I]\,[A-(n-r+nr)I]=0.$$

Hence find the inverse of the n x n matrix $B = [b_{ij}]$
Where $b_{ii} = 1$, $b_{ij} = \rho$ when $i \neq j$ and

$$\rho \neq 1,\ \rho \neq \frac{1}{1-n}.$$

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(b) Prove that the eigen vectors corresponding to distinct eigen values of a square matrix are linearly independent.

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(c) Determine the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

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3. (a) Show that a matrix congruent to a skew-symmetric matrix is skew-symmetric. Use the result to prove that the determinant of skew-symmetric matrix of even order is the square of a rational function of its elements.

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(b) Find the rank of the matrix

$$\begin{bmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{bmatrix}$$

where aa'+bb'+cc'=0 and a,b,c are all positive integers.

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(c) Reduce the following symmetric matrix to a diagonal form and interpret the result in terms of quadratic forms:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

4. (a) f(x) is defined as follows :

$$f(x) = \begin{vmatrix} \frac{1}{2}\left(b^2 - a^2\right) & \text{for} \quad 0 < x \le a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x} & \text{for} \quad a < x \le b \\ \frac{1}{3}\frac{b^3 - a^3}{x} & \text{for} \quad x > b \end{vmatrix}$$

Prove that f(x) and f'(x) are continuous but f''(x) is discontinuous.

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(b) If α and β lie between the least and greatest values of a, b, c prove that

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix} = K \begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \phi(a) & \phi'(\alpha) & \phi''(\beta) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta) \end{vmatrix}$$

Where K = 1/2 (b-c)(c-a)(a-b).

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(c) Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface.

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3. (a) Show by means of beta function that

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{\alpha}} = \frac{\pi}{\sin \pi\alpha}(0 < \alpha < 1)$$

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(b) Prove that the value of

$$\iiint \frac{dxdydz}{(x+y+z+1)^3}$$

taken over the volume bounded by the co-ordinate planes and the plane x+y+z=1, is

$$\frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

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(c) The sphere $x^2+y^2+z^2=a^2$ is pierced by the cylinder $(x^2+y^2)^2 = a^2(x^2-y^2)$. Prove by the cylinder $(x^2+y^2)^2=a^2(x^2-y^2)$. Prove that the volume of the sphere that lies inside the cylinder is

$$\frac{8a^3}{3}\left[\frac{\pi}{4} + \frac{5}{3} = \frac{4\sqrt{2}}{3}\right].$$

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6. (a) If 2ϕ be the angle between the tangents from

$P(x_1, y_1)$ to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that

$\lambda_1 \cos^2\phi + \lambda_2 \sin^2\phi = 0$ where λ_1, λ_2, are the parameters of two confocals to the ellipse through P.

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(b) If the normals at the points $\alpha, \beta, \gamma, \delta$ on the conic $1/r = 1+e\cos\theta$ meet at (ρ, ϕ), prove that $\alpha+\beta+\gamma+\delta-2\phi$=odd multiple of π radians.

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(c) A variable plane is at a constant distance p from the origin O and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$.

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7. (a) Find the equations to the generators of hyperboloid, through any point of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, z=0$.

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(b) Planes are drawn through a fixed point (α, β, γ) so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone.

$$\frac{\left(x-\alpha^2\right)}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0$$

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(c) Find $f(\theta)$ so that the curve $x = a\cos\theta$, $y = a\sin\theta$, $z = f(\theta)$ determines a plane curve.

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8. (a) Solve : $\frac{dy}{dx}+x\sin 2y = x^3\cos^2 y$

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(b) Show that if $\frac{1}{Q}\left[\frac{\partial p}{\partial y}-\frac{\partial Q}{\partial x}\right]$ is a function of x only, say, f(x), then $F(x)=e^{\int f(x)dx}$ is an integrating factor of $P\,dx + Q\,dy = 0$.

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(c) Find the family of curves whose tangent from an angle $\frac{\pi}{4}$ with the hyperbola xy=c.

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9. (a) Transform the differential equation

$$\frac{d^2y}{dx^2}\cos x+\frac{dy}{dx}\sin x-2y\cos^3 x=2\cos^5 x$$

into one having z as independent variable where $z = \sin x$ and solve it.

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(b) If $\frac{d^2x}{dt^2}+\frac{g}{b}(x-a)=0$ (a, b and g being positive constants) and $x = a'$ and $\frac{dx}{dt}=0$ when t=0, show that $x=a+(a'-a)\cos\sqrt{\frac{g}{b}}t$

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(c) Solve $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$

where $D \equiv \frac{d}{dx}$.

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10. (a) Show that $r^n \vec{r}$ is an irrotational vector for any value of n, but is solenoidal only if n = -3.

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(b) If $\vec{F} = y\vec{i} + (x-2xz)\vec{j} - xy\vec{k}$, evaluate

$\iint_S (\Delta \times \vec{F}).\vec{n}dS$

Where S is the surface of the sphere

$x^2+y^2+z^2=a^2$ above the xy plane.

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(c) Prove that $\left\{ \begin{matrix} i \\ ik \end{matrix} \right\} = \frac{\partial}{\partial x_k}\left(\log\sqrt{g}\right)$.

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11. (a) Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left\{ \frac{3}{\log\left(2+\sqrt{3}\right)} + \frac{4\pi}{3} \right\}.$$

20

(b) A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal and rests with its lower end on the quadrant of the curve which is farthest removed from the focus. Find its position of equilibrium and show that its length must at least be $\left(\frac{3a}{4} + \frac{a}{4}\sqrt{1+8e^2}\right)$ where 2a is the major axis and e, the eccentricity.

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(c) The height of a balloon is calculated from the barometric pressure reading (p) on the assumption that the pressure of the ai varies as the density. Show that if the pressure actually varies as the n^{th} power of the density, there will be an error

$$h_o\left[\frac{n}{n-1}\left\{ 1-\left(\frac{p}{p_o}\right)^{\frac{n-1}{n}} \right\} - \log\frac{p}{p_o} \right].$$

in the calculated height where h_o is the height of the homogeneous atmosphere and p_o is the pressure at the surface of the earth.

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12. (a) If in a simple Harmonic Motion, the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, be u, v, w respectively, show that the periodic time T is given by $\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

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(b) A gun is firing from the sea-level out to sea. It is mounted in a battery h meters high up and fired at the same elevation α. Show that the range is increased by $\frac{1}{2}\left[\left(1+\frac{2gh}{u^2\sin^2\alpha}\right)^{1/2} - 1\right]$ of itself, u being the velocity of projectile.

20

(c) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right]$ where V is the terminal velocity of the particle and λV is the initial vertical velocity.

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MATHEMATICS

Time Allowed: 3 hours **Maximum Marks: 300**

Candidates should attempt any five Questions.
ALL Questions carry equal marks.

PAPER-I

SECTION - A

1. (a) If G is a group such that $(ab)^n = a^n b^n$ for three consecutive integers n for all a,b in G, then prove that G is abelian.

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(b) Can a group of order 42 be simple ? Justify your claim.

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(c) Show that the additive group of integers modulo 4 is isomorphic to the multiplicative group of the non-zero elements of integers modulo 5. State the two isomorphisms.

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2. (a) Find all the units of the integral domain of Gaussian integers.

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(b) Prove or disprove the statement : The polynomial ring I [x] over the ring of integers is a principal ideal ring.

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(c) If R is an integral domain (not necessarily a unique factorization domain) and F is its field of quotients, then show that any element f(x) in F(x) is of the form $f(x) = \frac{f_0(x)}{a}$ where $f_0(x) \in R[x]$, $a \in R$.

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3. (a) If d is a metric on a nonempty set X, then is the function e defined by
$e(a,b) = \min(1, d(a,b)$
for $a,b \in X$, also a metric on X? Substantiate your claim.

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(b) Let C[0,1] denote the collection of all real continuous functions defined on I = [0,1]. Suppose d and e are metrics on C [0,1] defined by
$d(f,g) = \sup \{|fx)-g(x)| : x \in I\}$
$e(f,g) = \int_0^1 | f(x) - g(x) dx |.$
Is the topology τ_d induced by d coarser then τ_e, the topology induced by e? Justify your answer.

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(c) Examine the
(i) absolute convergence
(ii) uniform convergence
Of the series
$(1-x)+x(1-x)+x^2(1-x)+\ldots$
in [-c, 1] where $0<c<1$.

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4. (a) Prove that

$$S(x) = \Sigma \frac{1}{n^p + n^q x^2}, p > 1.$$

is uniformly convergent for all values of x and can be differentiated term by term if q<3p-2.

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(b) Let the function f be defined on [0,1] by the condition f(x) = 2 rx when $\frac{1}{r+1} < x < \frac{1}{r}, r > 0$.

Show that f is Riemann integrable in [0,1] and

$$\int_0^1 f(x)\,dx = \frac{\pi^2}{6}.$$

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(c) By means of the substitution

x+y+z=u, y+z=uv, z = uvw

evaluate $\iiint (x+y+z)^n \, xyz \, dx\, dy\, dz$

taken over the volume bounded by x=0, y=0, z=0, x+y+z=1.

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5. (a) Suppose that z is the position vector of a particle moving on the ellipse C :

z = a cos ωt + ib sin ωt.

where a, b, ω are positive constants, a > b and t is the time. Determine where

(i) the velocity has the greatest magnitude.

(ii) the acceleration has the least magnitude.

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(b) How many zeros does the polynomial

$$p(z) = z^4 + 2z^3 + 3z + 4$$

possess in (i) the first quadrant, (ii) the fourth quadrant.

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(c) Test of uniform convergence in the region |z|≤1 the series

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$$

6. (a) Find Laurent series for

(i) $\frac{e^{2z}}{(z-1)^3}$ about z = 1,

(ii) $\frac{1}{z^2(z-3)^2}$ about z = 3.

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(b) Find the residues of f(z) = $e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.

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(c) By means of contour integration , evaluate

$$\int_0^{\infty} \frac{(\log_e u)^2}{u^2+1} du.$$

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7. (a) Find the differential equation of the family of all cones with vertex at (2, -3, 1).

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(b) Find the integral surface of

$$x^2 p + y^2 q + z^2 = 0, p \equiv \frac{\partial z}{\partial x}, q \equiv \frac{\partial z}{\partial y}$$

which passes through the hyperbola

$xy = x+y,\ z = 1.$

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(c) Obtain a computer solution of $pq = x^m y^n z2l$.

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8. (a) Use Charpit's method to solve

$$16\,p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0.$$

Interpret geometrically the complete solution and mention the singular solution.

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(b) Solve $(D^2 + 3\,DD' + 2\,D'^2)z = x+y$, by expanding the particular integral in ascending powers of D, as well as in ascending powers of D'.

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(c) Find a surface satisfying $(D^2 + DD')z = 0$ and touching the elliptic paraboloid $z = 4x^2 + y^2$ along its section by the plane $y = 2x + 1$.

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SECTION - B

9. (a) What is D' Alembert's principle ? An inextensible string of negligible mass hanging over a smooth page at A connects the mass m_1 on a frictionless inclined plane of angle θ to another mass m^2. Use D' Alembert's principle to prove that the masses will be in equilibrium if $m_2 = m_1 \sin\theta$.

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(b) Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates of the system ? Write down Lagrange's equations of motion and obtain a first integral of the equations of motion.

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(c) Twelve equal uniform rods are smoothly joined at their ends so as to form a cubical framework, which is suspended from a point by a string tied to one corner and kept in shape by a light string occupying the position of a vertical diagonal. Suppose that the string supporting the framework is cut, so that it falls and strikes a smooth inelastic horizontal plane. Find the impulsive reaction of the plane.

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10. (a) A small light ring is threaded on a fixed thin horizontal wire. One end of a uniform rod, of mass m and length 2a is freely attached to the ring. The coefficient of friction between the ring and the wire is μ. The system is released from rest when the rod is horizontal and in the vertical plane containing the wire. If the ring slips on the wire when the rod has turned through an angle α, then prove that

$$\mu(10\tan^2\alpha + 1) = 9\tan\alpha.$$

30

(b) A uniform rod AB held at an inclination α to the vertical with one end A in contact with a rough horizontal table. If released, then prove that the rod will commence to slide at once if the coefficient of the friction μ is less than

$$\frac{3\sin\alpha\cos\alpha}{1+3\cos^2\alpha}.$$

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11. (a) The particle velocity for a fluid motion referred to rectangular axes is given by

$$\left(A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}, 0, A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right)$$

where A, a are constants. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube

$-a \le x \le a,\ 0 \le z \le 2a.$

Also find the pressure associated with this velocity field.

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(b) Determine the streamlines and the path lines of the particles when the velocity field is given by

$$\left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right).$$

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(c) Between the fixed boundaries $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, there is a two dimensional liquid motion due to a source of strength m at the point r = a, θ = 0 and an equal sink at the point r = b, q = 0. Use the method of images to show that the stream function is

$$-m\tan^{-1}\left\{\frac{r^4\left(a^4-b^4\right)\sin 4\theta}{r^8-r^4\left(a^4+b^4\right)\cos 4\theta+a^4b^4}\right\}$$

Show also that the velocity at (r, θ) is

$$\frac{4m\left(a^4-b^4\right)r^3}{\left(r^8-2a^4r^4\cos 4\theta+a^8\right)^{1/2}\left(r^8-2b^4r^4\cos 4\theta+b^8\right)^{1/2}}$$

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12. (a) Find the positive root of the equation

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}e^{0.3x}$$

correct to five decimal places.

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(b) Fit the following four points by the cubic splines.

i	0	1	2	3
x_i	1	2	3	4
y_i	1	5	11	8

Use the end conditions $y''_0 = y''_3 = 0$.

Hence compute (i) $y(1.5)$

(ii) $y'(2)$

20

(c) Find the derivative of f(x) at x = 0.4 from the following table :

x	0.1	0.2	0.3	0.4
y=f(x)	1.10517	1.22140	1.34986	1.49182

13. (a) Heights of 100 students at a college are as follows :

Height (in units)	***Number of students***
60-62	5
63-65	18
66-68	42
69-71	27
72-74	8

[1 unit = 2.5 cm]

Find the moment coefficient of skewness. Also obtain its value when Sheppard's corrections for grouping are used.

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(b) The frequency distributions of the final grades of 100 students in Mathematics and Physics are shown in the following table:

Mathematics grades

		40-49	**50-59**	**60-69**	**70-79**	**80-89**	**90-99**	**Total f_Y**
Physics grades	**90-99**				2	4	4	10
	80-89			1	4	6	5	16
	70-79			5	10	8	1	24
	60-69	1	4	9	5	2		21
	50-59	3	6	6	2			21
	40-49	3	5	4				12
	Total f_X	7	15	25	23	20	10	100

For this data, compute the standard error of estimate $S_{X,\,Y}$ and their covariance S_{XY}.

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(c) A college entrance examination consisted of 3 tests in Mathematics, English and General Knowledge. To test the ability of the examination to predict performance in a Statistics course, data concerning a sample of 200 students were gathered and analyzed. Letting

X_1 = grade in Statistics course

X_2 = score on Mathematics test

X_3 = score on English test

X_4 = score on General Knowledge test

the following calculations were obtained :

the mean of

$X_1 = \overline{X}_1 = 75, \overline{X}_2 = 24, \overline{X}_3 = 15, \overline{X}_4 = 36$;

the standard deviation $X_1 = s_1 = 10$, $s_2 = 5$, $s_3 = 3$, $s_4 = 6$;

the linear correlation coefficient between X_1 and $X_2 = r_{12} = 0.90$, $r_{13} = 0.75$, $r_{14} = 0.80$, $r_{23} = 0.70$, $r_{24} = 0{,}70$, $r_{34} = 0.85$.

Find the least square regression equation of X_1 and X_2, X_3 and X_4. What is the predicted grade in Statistics of a student who scores 30 in mathematics, 18 in English and 32 in general knowledge?

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14. (a) The contents of urns 1,2,3 are as follows :

1 white, 2 black, 3 red balls

2 white, 1 black, 1 red balls

4 white, 5 black, 3 red balls

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they came from urn 2 or 3?

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(b) Ten percent of the tools produced in a certain manufacturing process turn out to the defective. Find the probability that in a sample of 10 tools chosen at random, exactly two will be defective by using

(i) the binomial distribution

(ii) the Poisson approximation to the binomial distribution.

20

(c) Show that in a 2 x 2 contingency table wherein the frequencies are

a	b
c	d

Chi-square (χ^2) calculated from independent frequencies is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(c+d)(b+d)(a+c)}$$ 20

15. (a) Maximise $z = 3x_1 + 2x_2 + 5x_3$ (by Simplex method)

Subject to $x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_2 \leq 420$

$x_1, x_2, x_3 \geq 0$ 20

(b) Consider the following data :

Destinations

		1	2	3	Capacities
	1	2	2	3	10
Sources	2	4	1	2	15
	3	1	3	x	40
	Demands	20	15	30	

The cost of shipment from third source to the third destination is not known. How many units should be transported from the sources to the destinations so that the total cost of transporting all the units to their destinations is a minimum?

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(c) Divide a quantity b into n parts so as to maximize their product. Let $f_n(b)$ denote the value. Show that $f_1(b) = b$, $f_n(b) = \max \{zf_{n-1}(b-z)\}$. $0 \leq z \leq b$

Hence find $f_n(b)$ and the division that maximises it.

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16. (a) Apply Wolfe's method for solving the quadratic programming problem

Maximise

$$Z_x = 4x_1 + 6x_2 - 2(x_1)^2 - 2x_1x_2 - 2(x_2)^2$$

subject to

$$x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0.$$

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(b) A car hire company has one car at each of five depots a, b, c, d, e. A customer requires a car in each town namely A, B, C, D, E. Distance (in kilometers between depots (origins) and towns (destinations) are given in the following distance matrix:

	a	**b**	**c**	**d**	**e**
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	54	34	70	80	105

How should cars be assigned to customers so as to minimize the distance travelled?

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(c) A person is considering to purchase a machine for his own factory. Relevant data about alternative machines are as follows:

	Machine A	Machine B	Machine C
Present investment(Rs.)	10,000	12,000	15,000
Total annual cost(Rs.)	2,000	1,500	1,200
Life (years)	10	10	10
Salvage Value (Rs.)	500	1,000	1,200

As an adviser to the buyer, you have been asked to select the best machine, considering 12 per cent annual rate of return. You are given that

(i) Single payment present worth factor @ 12% for 10 years = 0.322.

(ii) Annual series present worth factor @ 12% for 10 years = 5.650.