MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt any FIVE questions.
All questions carry equal marks.

1. (a) Show that the set S = {(1,0,0), (1,1,0), (1,1,1), (0,1,0)}spans the vector space $R^3(R)$ but it is not a basis set.

   (b) Define rank and nullity of a linear transformation T. If V be a finite dimensional vector space and T a linear operator on v such that rank T^2 = rank T, then prove that the null space of T = the null space of T^2 and the intersection of the range space and null space to T is the zero subspace of V.

   (c) If the matrix of a linear operator T on R^2 relative to the standard basis {(1,0), (0,1)} is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, what is the matrix of T relative to the basis B = {(1,1), (1,-1)} ?

2. (a) Prove that the inverse of $\begin{pmatrix} A & O \\ B & C \end{pmatrix}$ is

$$\begin{pmatrix} A^{-1} & O \\ C^{-1}BA^{-1} & C^{-1} \end{pmatrix}$$

   where A, C are non-singular matrices and hence find the inverse of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

   (b) If A be an orthogonal matrix with the property that-1 is not an eigen value, then show that a is expressible as (I-S) $(S+S)^{-s1}$ for some suitable skew-symmetric matrix S.

   (c) Show that any two eigen vectors corresponding to two distinct eigen values of
   (i) Hermitian matrix
   (ii) Unitary matrix are orthogonal.

3. (a) A matrix B or order n x n is of the form λA where λ is a scalar and A has unit elements everywhere except in the diagonal which has elements μ. Find λ and μ so that B may be orthogonal.

   (b) Find the rank of the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix}$$

   by reducing it to canonical form.

   (c) Determine the following form as definite, semi-definite or indefinite:

$$2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_3x_1 + 2x_1x_2$$

4. (a) Prove that $f(x) = x^2 \sin \frac{1}{x}$, $x \neq 0$ and $f(x) = 0$ for $x = 0$ is continuous and differentiable at $x = 0$ but its derivative is not continuous there.

(b) If $f(x)$, $\phi(x)$, $\Psi(x)$ have derivatives when $a \leq x \leq b$, show that there is a value c of x lying between a and b such that

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(c) & \phi(c) & \psi(c) \end{vmatrix} = 0 \text{ ..f}$$

(c) Find the triangle of maximum area which can be inscribed in a circle.

5. (a) Prove that $\int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} (a > 0)$ and deduce that $\int_0^\infty x^{2n} e^{-x^2} dx = \frac{\sqrt{\pi}}{2^{n+1}} \left[1.3.5...(2n-1)\right]$.

(b) Define Gamma function and prove that

$$\lceil n \lceil \left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}} \lceil 2n$$

(c) Show that the volume common to the sphere $x^2+y^2+z^2 = a^2$ and the cylinder $x^2+y^2=ax$ is $\frac{2a^3}{9}(3\pi - 4)$.

6. (a) If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of lines, prove that the area of the triangle formed by their bisectors and axis of x is

$$\sqrt{\frac{(a-b)^2 + 4h^2}{2h}}, \qquad \frac{ca - g^2}{ab - h^2}$$

(b) Find the equation of the director circle of the conic $l/r = 1 + e \cos \theta$ and also obtain the asymptotes of the above conic.

(c) A line makes angles α, β, γ, δ with the diagonals of a cube. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$$

7. (a) Prove that the centres of the spheres which touch the lines $y = mx$, $z = c$; $y = -mx$, $z = -c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0$.

(b) Find the locus of the point of intersection of perpendicular generator of a hyperboloid of one sheet.

(c) A curve is drawn on a parabolic cylinder so as to cut all the generators at the same angle. Find its curvature and torsion.

8. (a) Determine the curvature for which the radius of curvature is proportional to the slope of the tangent.

(b) Show that the system of confocal conics

$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ is self-orthogonal.

(c) Solve $\{y(1+1/x)+\cos y\}\, dx + \{x+\log x - x \sin y\}dy=0$

9. (a) Solve $y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = y^2$

(b) Solve $\frac{d^2y}{dt^2} + \omega_0^2 y = a \cos \omega t$ and discuss the nature of solution as $\omega \xrightarrow{dt^2} \omega_0$.

(c) Solve $\left(D^4 + D^2 + 1\right) y e^{-x/2} \cos\left(x\sqrt{3}/2\right)$.

10. (a) Prove that the angular velocity of rotation at any point is equal to one half of the curl of the velocity vector V.

(b) Evaluate $\iint_s \Delta \times \vec{F} \hat{n}\, dS$ where S is the upper half surface of the unit sphere $x^2+y^2+z^2=1$ and $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$.

(c) Show that $\dfrac{\partial A_p}{\partial x^q}$ is not a tensor even though A_p is a covariant tensor or rank one.

11. (a) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log\left(\frac{1+\sqrt{1+\mu^2}}{\mu}\right)$$

where μ is the coefficient of friction.

(b) A solid hemisphere is supported by a string fixed to a point on its rim and to point on a smooth vertical wall with which the curved surface of the sphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$.

(c) A semi circular lamina is completely immersed in water with its plant vertical, so that the extremity A of its bounding diameter is in the surface and the diameter makes with the surface and angle α. Prove that if E be the centre of pressure and ϕ the angle between AE and the diameter,

$$\tan\phi = \frac{3\pi + 16\tan\alpha}{16 + 15\pi\tan\alpha}$$

12. (a) A point executes simple harmonic motion such that in two of its positions, the velocities are u and v and the corresponding accelerations are α and β. Show that the distance between the positions is $\dfrac{v^2 - u^2}{\alpha - \beta}$.

(b) A particle moves under a force

$$m\mu\{3au^4 - 2(a^2-b^2)u^5\},\ a>b$$

and is projected form an apse at a distance a + b with velocity $\dfrac{\sqrt{\mu}}{a+b}$. show that its orbit is r = a + b cosθ.

(c) A particle is projected upwards with a velocity u in a medium whose resistance varies as the square of the velocity. Prove that it will return to the point of projection with velocity $v = \dfrac{uV}{\sqrt{u^2+V^2}}$ after a time $\dfrac{V}{g}\left(\tan^{-1}\dfrac{u}{V} \tanh^{-1}\dfrac{u}{V}\right)$ where V is the terminal velocity.

MATHEMATICS

Time Allowed: 3 hours | ***Maximum Marks: 300***

Candidates should attempt any five Questions.
ALL Questions carry equal marks.

PAPER-I

SECTION - A

1. (a) If G is a cyclic group of order n and p divides n, then prove that there is a homomorphism of G onto a cyclic group of order p. what is the Kernel of homorphism?

   (b) Show that a group or order 56 cannot be simple.

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   (c) Suppose the H, K are normal subgroups of a finite group G with H a normal subgroup of K. If P = K/H, S = G/H, then prove that the quotient groups S/P and G/K are isomorphic.

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2. (a) If Z is the set of integers then show that

   $Z\left[\sqrt{-3}\right]=\left\{a+\sqrt{-3}b : a,b \in Z\right\}$

   is not a unique factorization domain.

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   (b) Construct the addition and multiplication table for

   $Z_3 [x] /< x^2 + 1>$

   where Z_3 is the set of integers modulo 3 and $<x^2+1>$ is the ideal generated by (x^2+1) in $Z_3[x]$.

   (c) Let Q be the set of rational number and Q($2^{1/2}$, $2^{1/3}$) the smallest extension field of Q containing $2^{1/2}$, $2^{1/3}$. Find the basis for Q($2^{1/2}$, $2^{1/3}$) over Q.

3. (a) Does the set C of all complex numbers form a metric space under

   $$d\left(z_1,z_2\right)=\frac{|z_1+z_2|}{\left\{\left(1+|z_1|^2\right)\left(1+|z_2|^2\right)\right\}^{1/2}}?$$

   Justify your claim.

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   (b) Show that x = 0 is a point of non-uniform convergence of the series whose nth term is $n^2 xe^{-n2x2}-(n-1)^2 x e^{-(n-1)}2x^2$.

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   (c) Find all the maxima and minima of

   $f(x,y) = x^3+y^3-63(x+y)+12xy$.

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4. (a) Examine for Riemann integrability over [0,2] of the function defined in [0,2] by

   $$f(x)=\begin{cases} x+x^2, \text{for rational values of x} \\ x^2+x^3, \text{for irrational values of x} \end{cases}$$

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   (b) Prove that $\int_0^{\infty}\frac{\sin x}{x}dx$ converges and conditionally converges.

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(c) Evaluate $\iiint \frac{dxdydz}{x+y+x+1}$

over the volume bounded by the coordinate planes and the plane x + y + z = 1.

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5. (a) In the finite z-plane, show that the function

$f(z) = \sec\frac{1}{z}$

has infinitely many isolated singularities in a finite intervals which includes 0.

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(b) Find the orthogonal trajectories of the family of curves in the xy-plane defined by

e-x(x sin y-y cos y)=α

where α is real constant.

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(c) Prove that (by applying Cauchy Integral formula or otherwise)

$$\int_0^{2\pi} \cos^{2n}\theta d\theta = \frac{1.3.5...(2n-1)}{2.4.6...2n}2\pi$$

where n = 1, 2, 3……….

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6. (a) If c is the curve $y = x^3-3x^2+4x-1$ joining the points (1,1) and (2,3) find the value of

$$\int_c \left(12z^2 - 4iz\right)dz$$

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(b) Prove that $\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)}$ converges absolutely for $|z|\leq 1$.

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(c) Evaluate

$$\int_0^{\infty}\frac{dx}{x^6+1}$$

by choosing an appropriate contour.

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7. (a) Find the surface whose tangent planes cut off an intercept of constant length R from the axis of z.

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(b) Solve $(x^3+3xy^2)p + (y^3+3x^2y)q=2(x^2+y^2)z$.

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(c) Find the integral surface of the partial differential equation (x-y)p+(y-x-z)q=z. thought the circle $z =1, x^2+y^2=1$.

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8. (a) Using Charpit's method find the complete integral of $2xz-px^2-2qxy+pq=0$.

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(b) Solve $r-s+2q-z=x^2y^2$.

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(c) Find the general solution of

$x^2r-y^2t+xp-yq = \log x$.

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SECTION – B

9. (a) Consider the two dynamical systems:

(I) A sphere rolling down from the top of a fixed sphere.

(II) A cylinder rolling without slipping down a rough inclined plane.

(i) State whether (I) is rheonomic, holonomic and conservative and justify your claim.

(ii) Give reasons for (II) to be classified as scleronomic, holonomic. Is it conservative?

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(b) Find

(i) the Lagrangian

(ii) the equations of motion

for the following system:

A particle is constrained to move in a plain under the influence of an attraction towards to origin proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction.

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(c) A heavy uniform rod rotating in a vertical plane falls and strikes a smooth inelastic horizontal plane. Find the impulse,

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10. (a) The door of a railway carriage has its hinges, supposed smooth, towards the engine, which starts with an acceleration f. Prove that the door closes in time.

$$\left(\frac{a^2+K^2}{2af}\right)^{1/2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$$

with an angular velocity $\sqrt{\frac{2af}{a^2+K^2}}$ where 2a is the breadth of the door and K its radius of gyration about a vertical axis through G, the centre of mass.

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(b) A solid homogeneous sphere is resting on the top of another fixed sphere and rolls down it. Write down the equations of motion and find the friction. When does the upper sphere leave the lower sphere if

(i) both the spheres are smooth

(ii) the upper sphere is sufficiently rough so as not to slip.

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11. (a) Show that

$$u=\frac{-2xyz}{\left(x^2+y^2\right)^2},v=\frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2},w=\frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion. Is this motion irrotational?

(b) Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if v and v be the corresponding velocities of the steam and if the motion be supposed to be that of divergence from the vertex of the cone, prove that

$$\frac{v}{V}=\frac{D2}{d^2}e\left(v^2-V^2\right)/2K$$

where K is the pressure divided by the density and supposed constant.

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(c) Prove that for liquid circulating irrotatinally in part of the plane between two non-intersecting circles, the curves of constant velocity are Cassini's ovals.

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12. (a) Find correct to 3 decimal places the two positive roots of $2e^x-3x^2=2.5644$.

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(b) Evaluate approximately $\int_{-3}^{3} x^4 dx$ Simpson's rule by taking seven equidistant ordinates. Compare it with the value obtained by using the trapezoidal rule and with exact value.

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(c) Solve $dy/dx = xy$
for x=1.4 by Runge-kutta method, initially x=1, y=2 (Take h=0.2).

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13. (a) The intelligence quotient of 480 student is as follows :

Class marks X	**Frequency f**
70	4
74	9
78	16
82	28
86	45
90	66
94	85
98	72
102	54
106	38
110	27
114	18
118	11
122	5
126	2

Find the moment coefficient of skewness for the intelligence quotient. It is skewed to the left or right?

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(b) If the correlation coefficient of zero order in a set of P variates were equal to p, show that every partial correlation of the s-th order is

$$\frac{p}{1+sp}$$

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(c) Fit a second degree parabola to the following data taking x as the independent variable:

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x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	0

14. (a) An event A is known to be independent of the event B, $B \cup C$ and $B \cap C$. Show that it is also independent of C.

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(b) If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determine the probability that out of 2000 individuals

(i) exactly 3 will suffer a bad reaction

(ii) more than 2 will suffer a bad reaction.

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(c) A machine has produced, in the past, washers having a thickness of 0.050 inches. To determine whether the machine is in proper working order, a sample of 10 washers is chosen for which the mean thickness is 0.053 inches and the standard deviation is 0.003 inches. Test the hypothesis that the machine is in proper working order using a level of significance of

(i) 0.05 (ii) 0.01.

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15. (a) Use Simplex method to solve:

Minimize $x_0 = x_1 - 3x_2 + 2x_3$

Subject to $3x_1-x_2+2x_3 \leq 7$

$-2x_1+4x_2 \leq 12$

$-4x_1+3x_2+8x_3 \leq 10$

$x_1,x_2,x_3 \leq 0.$

(b) A Departmental Head has four subordinates and four tasks are to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimates of the times each man would take to perform each task is given in the effectiveness matrix below. How should the tasks be allocated one to one man, so as to minimize the total man hours?

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Man

Task	I	II	III	IV
A	8	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

(c) Minimize $z = (y_1)^2 + (y_2)^2 + (y_3)^2$

Subject to $y_1 + y_2 + y_3 \geq 15,$

$y_1, y_2, y_3 \geq 0,$

by dynamic programming.

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16. (a) Use Kuhn-Tucker conditions to

Minimize $f(x) = (x_1)^2 + (x_2)^2 + (x_3)^2$

subject to $2x_1 + x_2 - 5 \leq 0$

$x_1 + x_3 - 2 \leq 0$

$1-x_1 \leq 0$

$2-x_2 \leq 0$

$-x_3 \leq 0$

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(b) On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patient ?

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(c) A transport manager finds from his past records that the costs per year of running a two-wheeler whose purchase cost is Rs. 6000/- are given below:

Year	**Running Costs**	**Resale (Salvage) price**
1	1000	3000
2	1200	1500
3	1400	750
4	1800	375
5	2300	200
6	2800	200
7	3400	200
8	4000	200

At what age the two wheeler should be replaced ? (Note that the money value is constant in time)

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