MATHEMATICS

Time Allowed: 3 hours **_Maximum Marks: 300_**

Candidates should attempt any FIVE questions.
All questions carry equal marks.

PAPER-I

1. (a) Let V and U be vector spaces over the field K and let V be of finite dimension. Let T: V. → U be a linear Map.
Prove that
dim V = dim R(T) + dim N(T)

(b) Let S={(x,y,z)/x+y+z=0}, x, y, z being real. Prove that S is a subspace of $\underset{\sim}{R}^3$. Find a basis of S.

(c) Verify which of the following are linear transformations:

(i) $T:\underset{\sim}{R}\to\underset{\sim}{R}^2$ defined by $T(x)=(2x,-x)$

(ii) $T:\underset{\sim}{R}^2\to\underset{\sim}{R}^3$ defined by $T(x,y)=(xy,y,x)$

(iii) $T:\underset{\sim}{R}^2\to\underset{\sim}{R}^3$ defined by $T(x,y)=(x+y,y,x)$

(iv) $T:\underset{\sim}{R}\to\underset{\sim}{R}^2$ defined by $T(x)=(1,-1)$

2. (a) Let T : $M_{2,1} \to M_{2,3}$ be a linear transformation defined by (with usual notations)

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2&1&3\\4&1&5\end{pmatrix},T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}6&1&0\\0&0&2\end{pmatrix}$$

Find $T\left(\dfrac{x}{y}\right)$.

(b) For what values of η do the following equations

$$x+y+z=1$$
$$x+2y+4z=\eta$$
$$x+4y+10z=\eta^2$$

Have a solution ? Solve them completely in each case.

(c) Prove that a necessary and sufficient condition of a real quadratic form X'AX to be positive definite is that the leading principal minors of A are all positive.

3. (a) State Cayley-Hamilton theorem and use it to calculate the inverse of the matrix→$\begin{bmatrix}2&1\\4&3\end{bmatrix}$

(b) Transform the following to the diagonal forms and give the transformation employed:
x^2+2y, $8x^2-4xy+5y^2$

(c) Prove that the characteristic roots of a Hermitian matrix are all real and a characteristic root of a skew-Hermitian matrix is either zero or a pure imaginary number.

4. (a) If $y=e^{ax}\cos bx$, prove that
$y_2-2ay_1+(a^2+b^2)y=0$
and hence expand $e^{2x}\cos bx$ in powers of x.
Deduce the expansion of e^{ax} and cos bx.

(b) If x = r sin θ cos ϕ, y = r sin θ sin ϕ, z = r cosθ
then prove that

$dx^2+dy^2+dz^2 = dr^2+r^2\, d\theta^2+r^2 \sin^2\theta\, d\phi^2$

(c) Find the dimensions of the rectangular parallelepiped inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ that has greatest volume.

5. (a) Prove that the volume enclosed by the cylinders
$x^2+y^2=2ax$, $z^2 = 2ax$ is $128a^3/15$

(b) Find the centre of gravity of the volume formed by revolving the area bounded by the parabolas $y^2=4ax$ and $x^2=4by$ about the x-axis.

(c) Evaluate the following integral in terms of Gamma function :

$$\int_{1}^{+1}(1+x)^p(1-x)^q\,dx, \qquad [p>-1, q>-1]$$

and prove that

$$\Gamma(1/3)\Gamma(2/3)=\frac{2}{\sqrt{3}}\pi .$$

6. (a) If $ax^2+2hxy+by^2+2gx+2fy+c=0$
represents two intersecting straight lines, show that the square of the distance of the point of intersection of the straight lines from the origin is

$$\frac{c(a+b)-f^2-g^2}{ab-h^2}\left(ab-h^2\neq 0\right)$$

(b) Discuss the nature of the conic
$16x^2-24xy+9y^2-104x-172y+144=0$
in details.

(c) A straight line, always parallel to the plane of yz, passed through the curves $x^2+y^2=a^2$, $z=0$, and $x^2=ax$, $y=0$; prove that the equation of the surface generated is
$x^4y^2=(x^2-az)^2(a^2-x^2)$

7. (a) Tangent planes are drawn to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

through the point (α,β,γ). Prove that the perpendiculars to them from the origin generate the cone

$(\alpha x,\beta y,\gamma z)^2 = a^2x^2+b^2y^2+c^2z^2$

(b) Show that the locus of the foot of the perpendicular from the centre to the plane through the extremities of three conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is

$a^2x^2 + b^2y^2 + c^2z^2 = 3(x^2+y^2+z^2)$

(c) Define an osculating plane and derive its equation in vector form. If the tangent and binormal at a point P of the curve make angles θ, ϕ respectively with the fixed direction, show that

$$\left(\frac{\sin\theta}{\sin\phi}\right)\left(\frac{d\theta}{d\phi}\right)=-\left(\frac{k}{\tau}\right)$$

where k and τ are respectively curvature and torsion of the curve at P.

8. (a) By eliminating the constants a, b obtain the differential equation of which $xy=ae^x+be^{-x}+x^2$ is a solution.

(b) Find the orthogonal trajectories of the family of the semi-cubical parabolas $ay^2=x^3$, where a is a variable parameter.

(c) Show that
$(4x+3y+1)dx+(3x+2y+1)dy=0$
represents hyperbolas having the following lines as asymptotes
$x+y=0$, $2x+y+1=0$

(d) Solve the following differential equation:

$$y(1+xy)dx+x(1-xy)dy=0$$

9. (a) Find the curves for which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact.

(b) Solve the following differential equation:

$(D^2+4)y=\sin 2x$, given that when x=0, then y=0 and $\frac{dy}{dx}=2$.

(c) Solve : $(D^3-1)y=xe^x+\cos^2 x$

(d) Solve : $(x^2D^2+xD-4)y=x^2$

10. (a) If $\vec{f}(x,y,z)=(y^2+z^2)\vec{i}+(z^2+x^2)\vec{j}+(x^2+y^2)\vec{k}$, then calculate $\int_c \vec{f}.\vec{dx}$, where C consists of

(i) the line segment from (0,0,0) to (1,1,1)

(ii) the three line segments AB, BC and CD, where A, B, C and D are respectively the points (0,0,0), (1,0,0), (1,1,0) and (1,1,1)

(iii) the curve $\vec{x}+\vec{ui}+\overrightarrow{u^2 j}+\overrightarrow{u^2 k}$, u from 0 to 1.

(b) If $\vec{a}$ and $\vec{b}$ are constant vectors, show that

(i) $div\{x\times(\vec{a}\times\vec{x})\}=-2\vec{x}.\vec{a}$

(ii) $div\{(\vec{a}\times\vec{x})\times(\vec{b}\times\vec{x})\}=2\vec{a}.(\vec{b}\times\vec{x})-2b.(\vec{a}\times\vec{x})$

(c) Obtain the formula

$$div\,\vec{A}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{-1}}\left\{\left(\frac{g}{g_{ii}}\right)^{1/2}A(i)\right\}$$

where A(i) are physical components of $\vec{A}$ and use it to derive expression of div $\vec{A}$ in cylindrical polar coordinates.

11. (a) Two equal rods, each of weight *wl* and length *l*, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal. Find the tension in the string and if the string is slack, show that ϕ satisfies the equation

$$\tan^3\phi+\tan\phi=\frac{1}{2r}$$

(b) Define central axis for a system of forces acting on a rigid body. A force F acts along the axis of x and another force nF along a generator of the cylinder $x^2+y^2=a^2$. Show that the central axis lies on the cylinder.

$$n^2(nx-z)^2+(1+n^2)^2y^2=n^4a^2$$

(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{3\pi(a^2+4b^2)+32ab}{4(3b\pi+4a)}$$

12. (a) A particle is moving with central acceleration $\mu(r^5-c^4r)$ being projected from an apse at a distance c with a velocity $\sqrt{\left(\frac{2\mu}{3}\right)}c^3$. Show that its path is the curve

$$x^4+y^4=c^4$$

(b) A particle is projected with a velocity whose horizontal and vertical components are respectively u and v from a given point in a medium whose resistance per unit mass is k times the speed. Obtain the equation of the path and prove that if k is small, the horizontal range is approximately

$$\frac{2uv}{g} - \frac{8uv^2k}{3g}$$

(c) A particle slides down the arc of a smooth vertical circle of radius a being slightly displaced from rest at the highest point of the circle. Find the point where it will strike the horizontal plane through the lowest point of the circle.

MATHEMATICS

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PAPER-I

SECTION - A

1. (a) If H is a cyclic normal subgroup of a group G, then show that every subgroup of H is normal in G.

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(b) Show that no group of order 30 is simple.

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(c) If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal.

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2. (a) If R is a unique factorization domain, then prove that any $f \in R[x]$ is an irreducible element of R[x], if and only if either f is an irreducible element of R or f is an irreducible polynomial in R[x}.

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(b) Prove that x^2+1 and x^2+x+4 are irreducible over F, the field of integers modulo 11. Prove also that $\frac{F[x]}{<x^2+1>}$ and $\frac{F[x]}{<x^2+x+4>}$ are isomorphic fields each having 121 elements.

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(c) Find the degree of splitting field $x^5-3x^3+x^2-3$ over Q, the field of rational numbers.

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3. (a) If we metrize the space of functions continuous on [a,b] by taking.

$$p(x,y)=\sqrt{\int_a^b /x(t)-y(t)/^2\, dt]}$$

then show that the resulting metric space is NOT complete.

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(b) Examine

$$f(x,y,z)=2xyz-4zx-2yz+x^2+y^2+z^2-2x-4y-4z$$

for extreme values.

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(c) If

$$U_n=\frac{1+nx}{ne^{nx}}-\frac{1+(n+1)x}{(n+1)e^{(n+1)x}}, \quad 0<x<1$$

Prove that

$$\frac{d}{dx}\Sigma U_n=\Sigma\frac{d}{dx}u_n$$

Is the series uniformly convergent in]0,1[? Justify your clain.

4. (a) Find the upper and lower Riemann integral for the function defined in the interval [0,1] as follows:

$$f(x)=\sqrt{1-x^2} \text{ when x is rational}$$
$$= 1\text{-x when x is irrational}$$

and show that f(x) is NOT Riemann integrable in [0,1].

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(b) Discuss the convergence or divergence of

$$\int_0^\infty \frac{x^\beta dx}{1+x\alpha\sin^2 x}, \qquad \alpha>\beta>0$$

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(c) Evaluate

$$\iint \frac{\sqrt{\left(a^2b^2-b^2x^2-a^2y^2\right)}}{\sqrt{\left(a^2b^2+b^2x^2+a^2y^2\right)}}dxdy$$

over the positive quadrant of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

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5. (a) If u=e^{-x}(x siny- $y$cosy), find v such that f(z)=u+iv is analytic. Also find f(z) explicitly as a function of z.

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(b) Let f(z) be analytic inside and on the circle C defined by |z|=R and let z=$er^{i\theta}$ be any point in side C. Prove that

$$f\left(re^{i\theta}\right)=1/2\pi\int_0^{2\pi}\frac{\left(R^2-r^2\right)f\left(\mathrm{Re}^{i\phi}\right)}{R^2-2Rr\cos\left(\theta+\phi\right)+r^2}d\phi$$

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(c) Prove that all the roots of

z^7-5z^3+12=0

lie between the circle |z| = 1 and |z|=2.

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6. (a) Find the region of convergence of the series whose n-th term is

$$\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$$

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(b) Expand

$$f(z)=\frac{1}{(z+1)(z+3)}$$

in a Laurent series valid for

(i) |z|>3

(ii) |<|z|<3

(iii) |z|<1

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(c) By integrating along a suitable contour evaluate

$$\int_0^8 \frac{\cos mx}{x^2+1}dx$$

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7. (a) Solve :

$(2x^2-y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z^2-yz-2zx-xy)q=x^2+y^2+2z^2-yz-zx-2xy$

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(b) Find the complete integral of

$$(y-x)(qy-px) = (p-q)^2$$

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(c) Use Charpit's method to solve

$$px+qy=z\sqrt{1+pq}$$

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8. (a) Find the surface passing through the parabolas

$z=0,\ y^2=4ax$

$z=1,\ y^2-4ax$

and satisfying the differential equation

$xr+2p=0$

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(b) Solve:

$r+s-6t=y\cos x$

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(c) Solve:

$$\frac{\partial^2 z}{\partial z^2}\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}-z=\cos\left(x+2y\right)+e^y$$

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9. (a) Classify each of the following dynamical systems according as they are

(i) scleronomic or rheonomic

(ii) holonomic or nonholonomic

(iii) conservative or non-conservative

[I] a horizontal cylinder of radius a rolling inside a perfectly rough hollow horizontal cylinder of radius b>a.

[II] a particle constrained to move along a line under the influence of a force which is inversely proportional to the square of its distance from a fixed point and a damping force proportional to the square of the instantaneous speed.

[III] a particle moving on a very long frictionless wire which rotates with constant angular speed about a horizontal axis.

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(b) When the Lagrangian function has the form

$L = q_k q_k - \sqrt{(1-q^2_K)}$

show that the generalized acceleration is zero.

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(c) The ends of a uniform rod AB of length 2a cos 15° and weight W are constrained to slide on a smooth circular wire of radius a fixed with its plane vertical. The end A is connected by an elastic string of natural length a and modulus of elasticity $W/_2$ to the highest point of the wire. If θ is the angle which the perpendicular bisector of the rod makes with the downward vertical, show that the potential energy V is given by

$$V = -\frac{W_a}{2}\left\{\cos\left(\theta - 75^o\right) + 2\cos{}^1\!/_2\left(\theta + 75^o\right)\right\} + \text{costant}$$

Verify that θ=25° defines a position of equilibrium and investigate its stability.

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10. (a) A uniform rod of length 2a which has one end attached to a fixed point by a light inextensible string of length 5a/12 is performing small oscillations in a vertical plane about its position of equilibrium. Find the position at any time and show that the periods of its principal oscillations are

$$2\pi\sqrt{\frac{5a}{3g}} \text{ and } \pi\sqrt{\frac{a}{3g}}.$$

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(b) a uniform circular disc of radius a and mass m rolls down a rough inclined plane without sliding. Show that the centre of the disc moves with constant acceleration 2/3 g sin α and the coefficient of friction μ > 1/3 tan α, where α is the inclination of the plane.

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11. (a) Show that the variable ellipsoid

$$\frac{x^2}{a^2k^2t^4} + kt^2\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] = 1$$

is a possible form for the boundary surface of a liquid motion at any time t.

(b) Find the lines of flow in the 2-dimensional fluid motion given by

$$\phi + i\Psi = -1/2n(x+iy)^2e^{2int}$$

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(c) A source of strength m and a vortex of strength k are placed at the origin of the 2-dimensional motion of unbounded liquid. Prove that the pressure at infinity exceeds that pressure at distance r from the origin by

$$\frac{1}{2}-\frac{\left(m^2+k^2\right)}{r^2}p$$

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12. (a) Compute to 4 decimal placed by using Newton-Raphson method, the real root of

$$x^2 + 4\sin x = 0$$

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(b) Solve by Runge-Kutta method

$$\frac{dy}{dx} = x + y$$

with the initial conditions $x_0=0$, $y_0=1$ correct up to 4 decimal places, by evaluating up to second increment of y. (Take h=0.1)

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(c) Fit the natural cubic spline for the data

x:	0	1	2	3	4
y:	0	0	1	0	0

13. (a) The standard deviations of two sets containing n_1 and n_2 are σ_1 and σ_2 respectively being measured from their respective means m_1 and m_2. If the two sets are grouped together as one set of $n_1 + n_2$ members, show that the standard deviation σ of this set measured from its mean is given by

$$\sigma^2 = \frac{n_1\sigma^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 + m_2)^2$$

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(b) Each coefficient in the equation $Ax^2+Bx+C=0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.

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(c) Prove that if A, B and C are random events in a sample space and if A, B, C are pair wise independent and A is independent of $B \cup C$, then A, B and C are mutually independent.

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14. (a) Fit the curve $y=ae^{bx}$ to the following data (e being napierian base 2.71828):

x :	0	2	4
y :	5.012	10	31.62

(b) Let x_1^2, x_2^2 be independently distributed variates having chi-square distributions with n_1, n_2 degrees of freedom respectively. Derive the distribution of

$$F = \frac{x_1^2 / n_1}{x_2^2 / n_2}$$

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(c) For a random sample of 10 horses fed on diet A, the increase in weight in kilograms in a certain period were

10, 6, 16, 17, 13, 12, 8, 14, 15, 9

For another random sample of 12 horses fed on diet B, the increase in the same period were

7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 kg

Test whether the diets A and B differ significantly as regard the effect on increase in weight. You may use the fact that 5% value of t for 20 degrees of freedom is 2.09.

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15. (a) Maximize $z=3x_1+2x_2$ subject to

$x_1+x_2 \le 4.$

$x_1 - x_2 \le 2.$

$x_1, x_2 \ge 0.$

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(b) the following table gives the cost for transporting material from supply points A,B,C,D to demand points E, F, G, H, J:

To

		E	F	G	H	J
From	A	8	10	12	17	15
	B	15	13	18	11	9
	C	14	20	6	10	13
	D	13	19	7	5	12

The present allocation is as follows :

A to E 90; A to F 10; B to F 150; C to F10

C to G 50; C to J 120; D to H 210; D to J 70

(i) Check if this allocation is optimum. If not, find an optimum schedule.

(ii) If in the above problem the transportation cost from A to G is reduce to 10, what will be the new optimum schedule?

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(c) Minimize $z = y_1 + y_2 + \ldots + y_n$ subject to $y_1 y_2 \ldots y_n = d$ and $y_j \ge 0$ for all j.

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16. (a) Determine x_1, x_2, x_3 so as to maximize

$$z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

subject to the constraints

$x_1+x_2 \le 2$

$2x_1+3x_2 \le 12$

$x_1, x_2 \ge 2$

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(b) At what average rate must a clerk at a super-market work in order to insure a probability of 0.90 that the customer will not have to wait longer than 12 minutes? It is assumed that there is only one counter to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

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(c) There are 5 jobs, each of which must go through machines A, B and C in the order ABC. Processing times are given in the following table:

Job	**Processing**	**times**	**(hours)**
	A	B	C
1	8	5	4
2	10	6	9
3	6	2	8
4	7	3	6
5	11	4	5

Determine a sequence for five jobs that will minimise the elapsed time.

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