UPSC Civil Services Main 2006 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{V}$ be a vector space of all 2×2 matrices over the field* F. *Prove that $\mathcal{V}$ has dimension* 4 *by exhibiting a basis for $\mathcal{V}$.*

Solution. Let $\mathbf{M_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{M_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{M_4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We will show that $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ is a basis of $\mathcal{V}$ over F.

$\underline{\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\} \text{ generate } \mathcal{V}}$. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{V}$. Then $\mathbf{A} = a\mathbf{M_1} + b\mathbf{M_2} + c\mathbf{M_3} + d\mathbf{M_4}$, where $a, b, c, d \in F$. Thus $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ is a set of generators for $\mathcal{V}$ over F.

$\underline{\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\} \text{ are linearly independent over } F}$. If $a\mathbf{M_1} + b\mathbf{M_2} + c\mathbf{M_3} + d\mathbf{M_4} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{0}$ for $a, b, c, d \in F$, then clearly $a = b = c = d = 0$, showing that $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ are linearly independent over F.

Hence $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ is a basis of $\mathcal{V}$ over F and $\dim \mathcal{V} = 4$. ∎

Question 1(b) *State the Cayley-Hamilton theorem and using it find the inverse of* $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Solution. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{I_n}$ be the $n \times n$ identity matrix. Then the n-degree polynomial $|\mathbf{x I_n} - \mathbf{A}|$ is called the charateristic polynomial of $\mathbf{A}$. The Cayley-Hamilton theorem states that every matrix is a root of its characteristic polynomial:

$$\begin{array}{ll} \text{if} & |\mathbf{x I_n} - \mathbf{A}| = x^n + a_1 x^{n-1} + \ldots + a_n \\ \text{then} & \mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \ldots + a_n \mathbf{I_n} = \mathbf{0} \end{array}$$

$|\mathbf{x I_n} - \mathbf{A}| = 0$ is called the characteristic equation of $\mathbf{A}$.

Let $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. The characteristic equation of $\mathbf{A}$ is $0 = \begin{vmatrix} x-1 & -3 \\ -2 & x-4 \end{vmatrix} = (x-1)(x-4) - 6 = x^2 - 5x - 2$.

By the Cayley-Hamilton Theorem, $\mathbf{A}^2 - 5\mathbf{A} - 2\mathbf{I_2} = \mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A} - 5\mathbf{I_2}) = (\mathbf{A} - 5\mathbf{I_2})\mathbf{A} = 2\mathbf{I_2}$. Thus $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A} - 5\mathbf{I_2})$, so $\mathbf{A}^{-1} = \frac{1}{2}\left[\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$ ∎

Question 2(a) *If* $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *is defined by* $\mathbf{T}(x, y) = (2x - 3y, x + y)$*, compute the matrix of* $\mathbf{T}$ *with respect to the basis* $\mathscr{B} = \{(1, 2), (2, 3)\}$.

Solution. It is obvious that $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation. Clearly

$$\begin{aligned} \mathbf{T}(\mathbf{v_1}) &= \mathbf{T}(1, 2) = (-4, 3) \\ \mathbf{T}(\mathbf{v_2}) &= \mathbf{T}(2, 3) = (-5, 5) \end{aligned}$$

Let $(a, b) = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, where $a, b, \alpha, \beta \in \mathbb{R}$, then $\alpha + 2\beta = a, 2\alpha + 3\beta = b \Rightarrow \alpha = 2b - 3a, \beta = 2a - b$. Thus $\mathbf{T}(\mathbf{v_1}) = 18\mathbf{v_1} - 11\mathbf{v_2}, \mathbf{T}(\mathbf{v_2}) = 25\mathbf{v_1} - 15\mathbf{v_1}$, so $(\mathbf{v_1}, \mathbf{v_2})\mathbf{T} = (\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2})) = (\mathbf{v_1}, \mathbf{v_2}) \begin{pmatrix} 18 & 25 \\ -11 & -15 \end{pmatrix}$. Thus the matrix of $\mathbf{T}$ with respect to the basis $\mathscr{B}$ is $\begin{pmatrix} 18 & 25 \\ -11 & -15 \end{pmatrix}$ ∎

Question 2(b) *Using elementary row operations, find the rank of* $\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$

Solution. Operations $\mathbf{R_1} - 2\mathbf{R_3}, \mathbf{R_2} - \mathbf{R_4}$ give

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Operation $\mathbf{R_3} - \mathbf{R_1}$ gives

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -9 & -5 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Operations $\mathbf{R_3} + 4\mathbf{R_2}, \mathbf{R_4} - \mathbf{R_2} \Rightarrow$

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$\mathbf{R_4} + \frac{2}{9}\mathbf{R_3} \Rightarrow$

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 0 & -\frac{1}{9} \end{pmatrix}$$

Clearly $|\mathbf{A}| = 1 \Rightarrow \operatorname{rank} \mathbf{A} = 4$. ∎

Question 2(c) *Investigate for what values of λ and μ the equations*

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

have (1) no solution (2) a unique solution (3) infinitely many solutions.

Solution. (2) The equations will have a unique solution for all values of μ if the coefficient matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{pmatrix}$ is non-singular. i.e. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & \lambda - 1 \end{vmatrix} = \lambda - 1 - 2 \neq 0$ i.e. $\lambda \neq 3$. Thus for $\lambda \neq 3$ and for all μ we have a unique solution which can be obtained by Cramer's rule or otherwise.

(1) If $\lambda = 3, \mu \neq 10$ then the system is inconsistent and we have no solution.

(3) If $\lambda = 3, \mu = 10$, the system will have infinitely many solutions obtained by solving $x + y = 6 - z, x + 2y = 10 - 3z \Rightarrow x = 2 + z, y = 4 - 2z$, z is any real number. ∎

Question 2(d) *Find the quadratic form $q(x, y)$ corresponding to the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix}$$

Is this quadratic form positive definite? Justify your answer.

Solution. The quadratic form is

$$\begin{aligned} q(x, y) &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= 5x^2 - 6xy + 8y^2 \\ &= 5[x^2 - \frac{6}{5}xy + \frac{8}{5}y^2] \\ &= 5[(x - \frac{3}{5}y)^2 + \frac{31}{25}y^2] \end{aligned}$$

Clearly $q(x, y) > 0$ for all $(x, y) \neq (0, 0), (x, y) \in \mathbb{R}^2$. Thus $q(x, y)$ is positive definite. In fact, $q(x, y) = 0 \Rightarrow x - \frac{3}{5}y = 0, y = 0 \Rightarrow x = y = 0$. ∎