UPSC Civil Services Main 2004 - Mathematics
Linear Algebra

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Let $\mathcal{S}$ be the space generated by the vectors $\{(0, 2, 6), (3, 1, 6), (4, -2, -2)\}$. What is the dimension of $\mathcal{S}$? Find a basis for $\mathcal{S}$.*

Solution. $(0, 2, 6), (3, 1, 6)$ are linearly independent, because $\alpha(0, 2, 6) + \beta(3, 1, 6) = \mathbf{0} \Rightarrow 3\beta = 0, 2\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$. Thus $\dim \mathcal{S} \geq 2$.

If possible let $(4, -2, -2) = \alpha(0, 2, 6) + \beta(3, 1, 6)$, then $4 = 3\beta, -2 = 2\alpha + \beta, -2 = 6\alpha + 6\beta$ should be consistent. Clearly $\beta = \frac{4}{3}, \alpha = \frac{1}{2}(-2 - \frac{4}{3}) = -\frac{5}{3}$ from the first two equations, and these values satisfy the third. Thus $(4, -2, -2)$ is a linear combination of $(0, 2, 6)$ and $(3, 1, 6)$.

Hence $\dim \mathcal{S} = 2$ and $\{(0, 2, 6), (3, 1, 6)\}$ is a basis of $\mathcal{S}$, being a maximal linearly independent subset of a generating system. ∎

Question 1(b) *Show that $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ where $f(x, y, z) = 3x + y - z$ is a linear transformation. What is the dimension of the kernel? Find a basis for the kernel.*

Solution.

$$\begin{aligned} f(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) &= f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \\ &= 3(\alpha x_1 + \beta x_2) + \alpha y_1 + \beta y_2 - (\alpha z_1 + \beta z_2) \\ &= \alpha(3x_1 + y_1 - z_1) + \beta(3x_2 + y_2 - z_2) \\ &= \alpha f(x_1, y_1, z_1) + \beta f(x_2, y_2, z_2) \end{aligned}$$

Thus f is a linear transformation.

Easy solution for this particular example. Clearly $(1, 0, 0)$ does not belong to the kernel, therefore the dimension of the kernel is ≤ 2. A simple look at f shows that $(0, 1, 1)$ and $(1, -1, 2)$ belong to the kernel and are linearly independent, thus the dimension of the kernel is 2 and $\{(0, 1, 1), (1, -1, 2)\}$ is a basis for the kernel.

General solution. Clearly $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is onto, thus the dimension of the range of f is 1. From question 3(a) of 1998, dimension of nullity of f + dimension of range of f = dimension of domain of f, so the dimension of the nullity of $f = 2$. Given this, we can pick a basis for the kernel by looking at the given transformation. ∎

Question 2(a) *Show that* $\mathbf{T}$ *the linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^4$ *represented by the matrix*

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

is one to one. Find a basis for its image.

Solution. $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis for $\mathbb{R}^3$. Then

$$\begin{aligned} \mathbf{T}(\mathbf{e_1}) &= (1, 0, 2, -1) = \mathbf{v_1} \\ \mathbf{T}(\mathbf{e_2}) &= (3, 1, 1, 1) = \mathbf{v_2} \\ \mathbf{T}(\mathbf{e_3}) &= (0, -2, 1, 2) = \mathbf{v_3} \end{aligned}$$

By linearity, if $\mathbf{T}(a, b, c) = a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = \mathbf{0}$, then $a + 3b = 0, b - 2c = 0, 2a + b + c = 0, -a + b + 2c = 0 \Rightarrow a = b = c = 0$. Thus $\mathbf{T}$ is one-one. Also $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ forms a basis for the image, since $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ generates $\mathbb{R}^3$, and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set. ∎

Question 2(b) *Verify whether the following system of equations is consistent:*

$$\begin{aligned} x + 3z &= 5 \\ -2x + 5y - z &= 0 \\ -x + 4y + z &= 4 \end{aligned}$$

Solution. The first equation gives $x = 5 - 3z$, the second now gives $5y = z + 10 - 6z = 10 - 5z \Rightarrow y = 2 - z$. Putting these values in the third equation we get $4 = -5 + 3z + 8 - 4z + z = 3$, hence the given system is inconsistent.

Alternative. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \\ -1 & 4 & 1 \end{pmatrix}$ be the coefficient matrix and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 3 & 5 \\ -2 & 5 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{pmatrix}$ be the augmented matrix, then it can be shown that rank $\mathbf{A} = 2$ and rank $\mathbf{B} = 3$, which implies that the system is inconsistent. For consistency the ranks should be equal. This procedure will be longer in this particular case. ∎

Question 2(c) *Find the characteristic polynomial of the matrix* $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$. *Hence find* $\mathbf{A}^{-1}$ *and* $\mathbf{A}^6$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by $|x\mathbf{I} - \mathbf{A}| = \begin{vmatrix} x-1 & -1 \\ 1 & x-3 \end{vmatrix} = (x-1)(x-3)+1 = x^2 - 4x + 4$.

The Cayley-Hamilton theorem states that $\mathbf{A}$ satisfies its characteristic equation i.e. $\mathbf{A}^2 - 4\mathbf{A} + 4\mathbf{I} = \mathbf{0} \Rightarrow (\mathbf{A} - 4\mathbf{I})\mathbf{A} = \mathbf{A}(\mathbf{A} - 4\mathbf{I}) = -4\mathbf{I}$. Thus $\mathbf{A}^{-1} = -\frac{\mathbf{A}-4\mathbf{I}}{4} = -\frac{1}{4}\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

From $\mathbf{A}^2 - 4\mathbf{A} + 4\mathbf{I} = \mathbf{0}$ we get

$$\begin{aligned}
\mathbf{A}^2 &= 4\mathbf{A} - 4\mathbf{I} \\
\mathbf{A}^3 &= 4\mathbf{A}^2 - 4\mathbf{A} = 4(4\mathbf{A} - 4\mathbf{I}) - 4\mathbf{A} = 12\mathbf{A} - 16\mathbf{I} \\
\mathbf{A}^6 &= (12\mathbf{A} - 16\mathbf{I})^2 = 144\mathbf{A}^2 - 384\mathbf{A} + 256\mathbf{I} = 144(4\mathbf{A} - 4\mathbf{I}) - 384\mathbf{A} + 256\mathbf{I} \\
&= 192\mathbf{A} - 320\mathbf{I} = \begin{pmatrix} -128 & 192 \\ -192 & 256 \end{pmatrix}
\end{aligned}$$

∎

Question 2(d) *Define a positive definite quadratic form. Reduce the quadratic form* $x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3$ *to canonical form. Is this quadratic form positive definite?*

Solution. If $\mathbb{Q}(x_1, \dots, x_n) = \sum_{\substack{i=1 \\ j=1}}^{n} a_{ij}x_ix_j, a_{ij} = a_{ji}$ is a quadratic form in n variables with $a_{ij} \in \mathbb{R}$, then it is said to be positive definite if $Q(\alpha_1, \dots, \alpha_n) > 0$ whenever $\alpha_i \in \mathbb{R}, i = 1, \dots, n$ and $\sum_i \alpha_i^2 > 0$.

Let the qiven be $\mathbb{Q}(x_1, x_2, x_3)$. Then

$$\begin{aligned}
\mathbb{Q}(x_1, x_2, x_3) &= x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 \\
&= (x_1 + x_2)^2 + x_3^2 + 2x_2x_3 - x_2^2 \\
&= (x_1 + x_2)^2 + (x_2 + x_3)^2 - 2x_2^2
\end{aligned}$$

Let $X_1 = x_1 + x_2, X_2 = x_2, X_3 = x_2 + x_3$ i.e.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

then $\mathbb{Q}(x_1, x_2, x_3)$ is transformed to $X_1^2 - 2X_2^2 + X_3^2$. Since $\mathbb{Q}(x_1, x_2, x_3)$ and the transformed quadratic form assume the same values, $\mathbb{Q}(x_1, x_2, x_3)$ is an indefinite form. The canonical form of $\mathbb{Q}(x_1, x_2, x_3)$ is $Z_1^2 - Z_2^2 + Z_3^2$ where $Z_1 = X_1, Z_2 = \sqrt{2}X_2, Z_3 = X_3$. ∎