UPSC Civil Services Main 2002 - Mathematics Linear Algebra

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Question 1(a) Show that the mapping $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ where $\mathbf{T}(a, b, c) = (a - b, b - c, a + c)$ is linear and non-singular.

Solution.

$$\mathbf{T}(\lambda a, \lambda b, \lambda c) = (\lambda(a-b), \lambda(b-c), \lambda(a+c))$$

= $\lambda(a-b, b-c, a+c)$
= $\lambda \mathbf{T}(a, b, c)$

$$\mathbf{T}(a,b,c) + \mathbf{T}(x,y,z) = (a-b,b-c,a+c) + (x-y,y-z,x+z) = (a-b+x-y,b-c+y-z,a+c+x+z) = \mathbf{T}(a+x,b+y,c+z)$$

Thus \mathbf{T} is linear.

Now we show that

$$\begin{array}{rcl} \mathbf{T}(1,0,0) &=& (1,0,1) \\ \mathbf{T}(0,1,0) &=& (-1,1,0) \\ \mathbf{T}(0,0,1) &=& (0,-1,1) \end{array}$$

are linearly independent.

$$a_1(1,0,1) + a_2(-1,1,0) + a_3(0,-1,1) = 0$$

$$\Rightarrow a_1 - a_2 = 0, a_2 - a_3 = 0, a_1 + a_3 = 0$$

$$\Rightarrow a_1 = a_2 = a_3 = 0$$

1 For more information log on www.brijrbedu.org. Copyright By Brij Bhooshan @ 2012. Thus (1, 0, 1), (-1, 1, 0), (0, -1, 1) are linearly independent.

Since (1, 0, 0), (0, 1, 0), (0, 0, 1) generate \mathbb{R}^3 , (1, 0, 1), (-1, 1, 0), (0, -1, 1) generate $\mathbf{T}(\mathbb{R}^3)$, hence dim $(\mathbf{T}(\mathbb{R}^3)) = 3$. Thus **T** is non-singular.

Alternatively,

$$\mathbf{T}(a,b,c) = (0,0,0) \iff a-b = 0, b-c = 0, a+c = 0 \implies a = b = c = 0$$

Thus \mathbf{T} is 1-1, therefore it is onto, which shows it is nonsingular.

Question 1(b) Prove that a square matrix \mathbf{A} is non-singular if and only if the constant term in its characteristic polynomial is different from 0.

Solution. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Then characteristic polynomial of $\mathbf{A} = \mathbf{Det}(x\mathbf{I} - \mathbf{A})$. $\mathbf{I} = n \times n$ unit matrix. Clearly

$$\mathbf{Det}(x\mathbf{I} - \mathbf{A}) = x^n - \sum_{i=1}^n a_{ii}x^{n-1} + \ldots + (-1)^n \mathbf{DetA}$$

Thus A is nonsingular iff the constant term in the characteristic polynomial of $\mathbf{A} \neq 0$.

Question 2(a) Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be a linear mapping given by T(a, b, c, d, e) = (b - d, d + e, b, 2d + e, b + e). Obtain bases for its null space and range space.

Solution. Clearly

T(1,0,0,0,0) = (0,0,0,0,0) T(0,1,0,0,0) = (1,0,1,0,1) T(0,0,1,0,0) = (0,0,0,0,0) T(0,0,0,1,0) = (-1,1,0,2,0)T(0,0,0,0,1) = (0,1,0,1,1)

are generators of the range space of T. In fact, if $\mathbf{v_1} = (1, 0, 1, 0, 1), \mathbf{v_2} = (-1, 1, 0, 2, 0), \mathbf{v_3} = (0, 1, 0, 1, 1)$ then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ generate $T(\mathbb{R}^5)$. We now show that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent. Let $\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \alpha_3 \mathbf{v_3} = 0$. Then $\alpha_1 - \alpha_2 = 0, \alpha_2 + \alpha_3 = 0, \alpha_1 = 0 \Rightarrow \alpha_2 = \alpha_3 = 0$. Thus $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent over $\mathbb{R} \Rightarrow T(\mathbb{R}^5)$ is of dimension 3 with basis $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

Thus the null space is of dimension 2, because dim(null space) + dim(range space) = dim(given vector space = \mathbb{R}^5) = 5. Since $\mathbf{e_1} = (1, 0, 0, 0, 0)$ and $\mathbf{e_3} = (0, 0, 1, 0, 0)$ belong to the null space of T, and both are linearly independent over \mathbb{R} , $\mathbf{e_1}$, $\mathbf{e_3}$ is a basis of the null space of T.

Question 2(b) Let \mathbf{A} be a 3 × 3 real symmetric matrix with eigenvalues 0, 0, 5. If the corresponding eigenvectors are (2,0,1), (2,1,1), (1,0,-2) then find the matrix \mathbf{A} .

Solution. Let
$$\mathbf{P} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$
, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, so $\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{P}^{-1}$.
A simple calculation shows that $\mathbf{P}^{-1} = \begin{pmatrix} \frac{2}{5} & -1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix}$, therefore

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$
Thus $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$ is the required symmetric matrix with 0, 0, 5 as eigenvalues.

Question 2(c) Solve the following system of linear equations:

$$x_1 - 2x_2 - 3x_3 + 4x_4 = -1$$

-x_1 + 3x_2 + 5x_3 - 5x_4 - 2x_5 = 0
2x_1 + x_2 - 2x_3 + 3x_4 - 4x_5 = 17

Solution. There are three equations in 5 unknowns, therefore the rank of the coefficient matrix ≤ 3 . Since $\begin{vmatrix} 1 & -2 & -3 \\ -1 & 3 & 5 \\ -2 & 1 & -2 \end{vmatrix} = 1(-6-5) + 2(2-10) + (-3)(-1-6) = -6$, the rank

of the coefficient matrix is 3. Using Cramer's rule we solve the system

$$x_1 - 2x_2 - 3x_3 = -1 - 4x_4 \tag{1}$$

$$-x_1 + 3x_2 + 5x_3 = 5x_4 + 2x_5 \tag{2}$$

$$2x_1 + x_2 - 2x_3 = 17 - 3x_4 + 4x_5 \tag{3}$$

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$$\begin{aligned} x_1 &= -\frac{1}{6} \begin{vmatrix} -1 - 4x_4 & -2 & -3 \\ 5x_4 + 2x_5 & 3 & 5 \\ 17 - 3x_4 + 4x_5 & 1 & -2 \end{vmatrix} \\ &= -\frac{1}{6} [(-1 - 4x_4)(-11) - 3(5x_4 + 2x_5 - 51 + 9x_4 - 12x_5) + 2(-10x_4 - 4x_5 - 85 + 15x_4 - 20x_5)] \\ &= -\frac{1}{6} [-6 + 44x_4 - 42x_4 + 10x_4 + 30x_5 - 48x_5] \\ &= 1 - 2x_4 + 3x_5 \end{aligned}$$

$$\begin{aligned} x_2 &= -\frac{1}{6} \begin{vmatrix} 1 & -1 - 4x_4 & -3 \\ -1 & 5x_4 + 2x_5 & 5 \\ 2 & 17 - 3x_4 + 4x_5 & -2 \end{vmatrix} \\ &= -\frac{1}{6} [-10x_4 - 4x_5 - 85 + 15x_4 - 20x_5 - 8 - 32x_4 + 51 - 9x_4 + 12x_5 + 30x_4 + 12x_5] \\ &= -\frac{1}{6} [-42 - 6x_4] \\ &= 7 + x_4 \end{aligned}$$

$$\begin{aligned} x_3 &= -\frac{1}{6} \begin{vmatrix} 1 & -2 & -1 - 4x_4 \\ -1 & 3 & 5x_4 + 2x_5 \\ 2 & 1 & 17 - 3x_4 + 4x_5 \end{vmatrix} \\ &= -\frac{1}{6} [51 - 9x_4 + 12x_5 - 5x_4 - 2x_5 - 34 + 6x_4 - 8x_5 - 20x_4 - 8x_5 + 7 + 28x_4] \\ &= -\frac{1}{6} [24 - 6x_5] \\ &= -4 + x_5 \end{aligned}$$

The solution space is $(1 - 2x_4 + 3x_5, 7 + x_4, -4 + x_5, x_4, x_5)$, where $x_4, x_5 \in \mathbb{R}$ (arbitrarily). Note that the vector space of solutions is of dimension 2.

Alternate Method.

$$x_2 + 2x_3 = -1 + x_4 + 2x_5$$
 adding (1) and (2) (4)

$$7x_2 + 8x_3 = 17 + 7x_4 + 8x_5$$
 adding $2 \times (2)$ and (3) (5)

$$6x_3 = -24 + 6x_5 \text{ using } 7 \times (4) - (5) \tag{6}$$

$$x_2 = 7 + x_4 \text{ from } (4) \text{ and } (6)$$
 (7)

$$x_1 = 1 - 2x_4 + 3x_5$$
 using (1), (6), (7) (8)

The solution space is as shown above.

4 For more information log on www.brijrbedu.org. Copyright By Brij Bhooshan @ 2012. Question 2(d) Use Cayley-Hamilton theorem to find the inverse of the following matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. Characteristic polynomial is given by $|x\mathbf{I} - \mathbf{A}| = 0$, where \mathbf{I} is the 3 × 3 unit matrix.

$$\begin{vmatrix} x & -1 & -2 \\ -1 & x - 2 & -3 \\ -3 & -1 & x - 1 \end{vmatrix} = 0$$
$$x[x^2 - 3x + 2 - 3] + 1[-x + 1 - 9] - 2[1 + 3x - 6] = 0$$
$$x^3 - 3x^2 - 8x + 2 = 0$$

By Cayley-Hamilton theorem, $\mathbf{A}^3 - 3\mathbf{A}^2 - 8\mathbf{A} + 2\mathbf{I} = 0$, or $\mathbf{A}(\mathbf{A}^2 - 3\mathbf{A} - 8\mathbf{I}) = -2\mathbf{I}$. Thus

$$\begin{aligned} \mathbf{A}^{-1} &= -\frac{1}{2} (\mathbf{A}^2 - 3\mathbf{A} - 8\mathbf{I}) \\ &= -\frac{1}{2} \left[\begin{pmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{pmatrix} - \begin{pmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} \right] \\ &= -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

Check $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.