UPSC Civil Services Main 1983 - Mathematics
Linear Algebra

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Question 1(a) *Let $\mathcal{V}$ be a finitely generated vector space. Show that $\mathcal{V}$ has a finite basis and any two bases of $\mathcal{V}$ have the same number of vectors.*

Solution. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a generating set for $\mathcal{V}$, we assume that $\mathbf{v}_i \neq \mathbf{0}, 1 \leq i \leq m$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent, then it is a basis of $\mathcal{V}$. Otherwise, there exists a $\mathbf{v}_k$ that depends linearly on $\{\mathbf{v}_i \mid 1 \leq i \leq m, i \neq k\}$. This latter set is also a generating set, and we rename it $\{\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}\}$. We now apply the same reasoning to it — either it is linearly independent and hence a basis, or we can drop an element from it and it still remains a generating set. In a finite number of steps, we reach $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is linearly independent and a generating set, thus $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis of $\mathcal{V}$.

Note: An alternative approach leading to the same result is to pick the maximal linearly independent subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. There are only 2^m such subsets, so we can do so in a finite number of steps (in the above procedure we dropped the dependent elements one at a time to reach the maximal linearly independent subset). Now to be a basis, the maximal linearly independent subset $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_r\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ needs to generate $\mathcal{V}$. But this is immediate, as for each $\mathbf{v}_i$, either $\mathbf{v}_i \in S$ or $S \cup \{\mathbf{v_i}\}$ is linearly dependent — in that case $\sum_{j=1}^r a_j \mathbf{x}_j + b\mathbf{v_i} = 0$, but not all a_j, b are 0. Now if $b = 0$ then $\sum_{j=1}^r a_j \mathbf{x}_j = 0 \Rightarrow a_j = 0$ for $1 \leq j \leq r$, as S is linearly independent, and this contradicts the statement that not all a_j, b are 0. So $b \neq 0$, hence $\mathbf{v}_i$ is a linear combination of S, hence S generates $\mathcal{V}$ and is a basis.

Any two bases have the same number of elements: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be two bases of $\mathcal{V}$. Assume wlog that $m \leq n$. Now since $\mathbf{w}_1 \in \mathcal{V}$, $\mathbf{w}_1$ is generated by the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, thus $\mathbf{w}_1 = \sum_{j=1}^m a_j \mathbf{v}_j$. There must be at least one non-zero a_k, as $\mathbf{w}_1 \neq \mathbf{0}$. Now the set $\{\mathbf{v}_i \mid 1 \leq i \leq m, i \neq k\} \cup \{\mathbf{w}_1\}$ generates the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ (since $\mathbf{v}_k = \frac{1}{a_k}\mathbf{w}_1 - \sum_{j=1, j\neq k}^m \frac{a_j}{a_k}\mathbf{v}_j$) and hence generates $\mathcal{V}$.

Now we have $\mathbf{w}_2 = \sum_{i=1, i\neq k}^m a_i \mathbf{v}_i + b\mathbf{w_1}$. At least one of the $a_i \neq 0$, otherwise we have a linear equation between $\mathbf{w}_1$ and $\mathbf{w}_2$, but these are linearly independent. We replace $\mathbf{v}_i$ by $\mathbf{w}_2$, and the result is also a generating set as above. Continuing, after m steps, we get a subset $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ which is a generating set. Now if $n > m$, we would have $\mathbf{w}_n = \sum_{i=0}^{m} a_i \mathbf{w}_i$, but this is not possible as the $\mathbf{w}_i$ were a basis, and thus linearly independent. Hence $n = m$, and the two bases have equal number of elements. ∎

Question 1(b) *Let $\mathcal{V}$ be the vector space of polynomials of degree ≤ 3. Determine whether the following vectors of $\mathcal{V}$ are linearly dependent or independent: $u = t^3 - 3t^2 + 5t + 1, v = t^3 - t^2 + 8t + 2, w = 2t^3 - 4t^2 + 9t + 5$.*

Solution. Let $au + bv + cw = 0$. Then

$$\begin{aligned} a + b + 2c &= 0 \quad (1) \\ -3a - b - 4c &= 0 \quad (2) \\ 5a + 8b + 9c &= 0 \quad (3) \\ a + 2b + 5c &= 0 \quad (4) \end{aligned}$$

From (4) - (1) we get $b + 3c = 0$. Substituting $b = -3c$ in (2), $c = -3a \Rightarrow b = 9a$. Now from (1), $a + 9a - 6a = 0 \Rightarrow a = 0 \Rightarrow b = c = 0$. Thus $au + bv + cw = 0 \Rightarrow a = b = c = 0$, so the vectors are linearly independent. ∎

Question 1(c) *For any linear transformation $T : \mathcal{V}_1 \to \mathcal{V}_2$ prove that*

$$\operatorname{rank} T \leq \min(\dim \mathcal{V}_1, \dim \mathcal{V}_2)$$

Solution. By definition, $\operatorname{rank} T = \dim T(\mathcal{V}_1)$. Clearly $T(\mathcal{V}_1)$ is a subspace of $\mathcal{V}_2$ and therefore $\dim T(\mathcal{V}_1) \leq \dim \mathcal{V}_2$. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of $\mathcal{V}_1$, then $T(\mathcal{V}_1)$ is generated by $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ — If $\mathbf{w} \in T(\mathcal{V}_1)$, then there exists $\mathbf{v} \in \mathcal{V}_1$ such that $T(\mathbf{v}) = \mathbf{w}$. But $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i, a_i \in \mathbb{R}$, therefore $\mathbf{w} = T(\mathbf{v}) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) \Rightarrow \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a generating system for $T(\mathcal{V}_1) \Rightarrow \dim T(\mathcal{V}_1) \leq n$. Thus $\operatorname{rank} T = \dim T(\mathcal{V}_1) \leq \min(\dim \mathcal{V}_1, \dim \mathcal{V}_2)$. ∎

Question 2(a) *Show that every non-singular matrix can be expressed as a product of elementary matrices.*

Solution. We first list all the elementary matrices:

1. $\mathbf{E}_{ij}$ = the matrix obtained by interchanging the i-th and j-th rows (or the i-th and j-th columns of the unit matrix. For example, if $n = 4$, then

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. $\mathbf{E}_i(\alpha)$ is the matrix obtained by multiplying the i-th row of the unit matrix by $\alpha =$ the matrix obtained by multiplying the i-th column of the unit matrix by α.

3. $\mathbf{E}_{ij}(\beta) =$ the matrix obtained by adding β times the j-th row to the i-th row of the unit matrix.

4. $(\mathbf{E}_{ij}(\beta))' =$ transpose of $\mathbf{E}_{ij}(\beta) =$ the matrix obtained by adding β times the j-th column to the i-th column of the unit matrix.

All elementary matrices are non-singular. In fact $|\mathbf{E}_{ij}| = -1$, $|\mathbf{E}_i(\alpha)| = \alpha$, $|\mathbf{E}_{ij}(\beta)| = |(\mathbf{E}_{ij}(\beta))'| = 1$.

We now prove the result.

(1) Let $\mathbf{C} = \mathbf{AB}$. Then any elementary row transformation on $\mathbf{AB}$ is equivalent to subjecting $\mathbf{A}$ to the same row transformation. Let $\mathbf{A} = \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix}_{m\times n}$ and $\mathbf{B} = \begin{pmatrix} C_1 & \dots & C_p \end{pmatrix}_{n\times p}$.

Then $\mathbf{AB} = \begin{pmatrix} R_1C_1 & \dots & R_1C_p \\ \vdots & & \vdots \\ R_mC_1 & \dots & R_mC_p \end{pmatrix}_{m\times p}$. Thus if any elementary row transformation i.e. (i) Intercanging two rows (ii) Multiplying a row by a scalar (iii) Adding a scalar multiple of a row to another row, is carried out on $\mathbf{A}$, the same will be carried out on $\mathbf{AB}$ and vice versa. Similarly any column transformation on $\mathbf{B}$ is equivalent to the same column transformation on $\mathbf{AB}$.

(2) Multiplying by an elementary matrix $\mathbf{E}_{ij}, \mathbf{E}_i(\alpha), \mathbf{E}_{ij}(\beta)$ on the left is the same as performing the corresponding elementary row operation on the matrix. Multiplying the matrix by an elementary matrix to the right is equal to subjecting the matrix to the corresponding column transformation. We write $\mathbf{A} = \mathbf{IA}$. Now interchanging the i-th and j-th row of $\mathbf{A}$ is equivalent to doing the same on $\mathbf{I}$ in $\mathbf{IA}$ (result (1) above), which is the same as $\mathbf{E}_{ij}\mathbf{A}$. Similar results hold for the other two row transformations. Writing $\mathbf{A}$ as $\mathbf{AI}$ gives the corresponding result for column transformations.

(3) We now prove that if $\mathbf{A}$ is a matrix of rank $r > 0$, then there exist $\mathbf{P}, \mathbf{Q}$ products of elementary matrices such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ where $\mathbf{I}_r$ is the unit matrix of order r. Since $\mathbf{A} \neq \mathbf{0}$, $\mathbf{A}$ has at least one non-zero element, say a_{ij}. By interchanging the i-th row with the first row and the j-th column with the first column, we get a new matrix $\mathbf{B}_{ij} = (b_{ij})$ such that $b_{11} \neq 0$. This simply means that there exist elementary matrices $\mathbf{P_1}, \mathbf{Q_1}$ such that $\mathbf{P_1AQ_1} = \mathbf{B}$. We multiply $\mathbf{P_1AQ_1}$ by $\mathbf{P_2} = \mathbf{E}_1(b_{11}^{-1})$ to obtain $\mathbf{P_2P_1AQ_1} = \mathbf{C} = \begin{pmatrix} 1 & * & \dots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$. Subtracting suitable multiples of the first row from the remaining rows of $\mathbf{C}$ and suitable multiples of the first column from the remaining columns, we get the new

matrix $\mathbf{D}$ of the form $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}^* & \\ 0 & & & \end{pmatrix}$. Thus we have proved that there exist $\mathbf{P}^*, \mathbf{Q}^*$ products of elementary matrices such that $\mathbf{P}^*\mathbf{A}\mathbf{Q}^* = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}^* & \\ 0 & & & \end{pmatrix}$. We carry on the same process on $\mathbf{A}^*$ without affecting the first row and column, and in r steps we get $\mathbf{P}^{**}\mathbf{A}\mathbf{Q}^{**} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}$, where $\mathbf{P}^{**}, \mathbf{Q}^{**}$ are products of elementary matrices. Note that $\mathbf{E} = \mathbf{0}$ because rank $\mathbf{A} = r$.

Now if $\mathbf{A}$ is nonsingular, then $\mathbf{P}^{**}\mathbf{A}\mathbf{Q}^{**} = \mathbf{I}$. Inverting the elementary matrices (the inverse of an elementary matrix is elementary), we get that $\mathbf{A}$ is a product of elementary matrices. ∎

Question 2(b) *Reduce the matrix* $\mathbf{A}$ *to its normal form, and hence or otherwise determine its rank.*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

Solution. Interchange of R_1 and $R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

$R_3 - 3R_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$

$R_3 + 5R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

$-\frac{1}{2}R_3 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$

$R_3 + R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Interchanging C_2, C_4, $\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$C_2 - 2C_1, C_3 - 3C_1, C_4 - 2C_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$C_3 - 2C_2, C_4 - C_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. $C_4 - C_3 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Thus we have $\mathbf{P}(3 \times 3)$ and $\mathbf{Q}(4 \times 4)$ both products of elementary matrices such that $\mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which is the normal form of $\mathbf{A}$. Clearly the rank of $\mathbf{A}$ is 3. ∎

Question 2(c) *Show that the equations*

$$\begin{aligned} x + y + z &= 3 \\ 3x - 5y + 2z &= 8 \\ 5x - 3y + 4z &= 14 \end{aligned}$$

are consistent and solve them.

Solution. The coefficient matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{pmatrix}$.

$\det \mathbf{A} = 1(-20+6) - 1(12-10) + 1(-9+25) = 0$, thus rank $\mathbf{A} < 3$. Actually rank $\mathbf{A} = 2$, since $\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} \neq 0$.

The augmented matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 3 & -5 & 2 & 8 \\ 5 & -3 & 4 & 14 \end{pmatrix}$.

$R_2 - 3R_1, R_3 - 5R_1 \Rightarrow \mathbf{B} \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & -8 & -1 & -1 \end{pmatrix}$

$R_3 - R_2 \Rightarrow \mathbf{B} \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus rank $\mathbf{B} = 2$, because $\begin{vmatrix} 1 & 1 \\ 0 & -8 \end{vmatrix} \neq 0$.

Since rank $\mathbf{A}$ = rank $\mathbf{B} = 2$, the system is consistent, and the space of solutions has dimension 1.

Now $x + y = 3 - z, 3x - 5y = 8 - 2z$, subtracting the second from 3 times the first we get $8y = 1 - z \Rightarrow y = \frac{1-z}{8}$. $x = 3 - z - \frac{1-z}{8} = \frac{23-7z}{8}$. Thus the solutions are given by $(\frac{23-7z}{8}, \frac{1-z}{8}, z)$, $z \in \mathbb{R}$. ∎

Question 3(a) *Prove that a square matrix satisfies its characteristic equation. Use this result to find the inverse of*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. The first part is the Cayley-Hamilton theorem, see 1987 question 3(a).

The characteristic equation of $\mathbf{A}$ is

$$\begin{aligned}|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix}-\lambda & 1 & 2\\ 1 & 2-\lambda & 3\\ 3 & 1 & 1-\lambda\end{vmatrix} &= 0\\ \Rightarrow -\lambda(\lambda^2-3\lambda+2-3)-(1-\lambda-9)+2(1-6+3\lambda) &= 0\\ \Rightarrow \lambda^3-3\lambda^2-8\lambda+2 &= 0\end{aligned}$$

Thus $\mathbf{A}^3-3\mathbf{A}^2-8\mathbf{A}+2\mathbf{I}=\mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A}^2-3\mathbf{A}-8\mathbf{I}) = -2\mathbf{I}$, or $\mathbf{A}^{-1} = -\frac{1}{2}(\mathbf{A}^2-3\mathbf{A}-8\mathbf{I})$.

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix}0&1&2\\1&2&3\\3&1&1\end{pmatrix}\begin{pmatrix}0&1&2\\1&2&3\\3&1&1\end{pmatrix} = \begin{pmatrix}7&4&5\\11&8&11\\4&6&10\end{pmatrix}\\ \therefore \mathbf{A}^{-1} &= \frac{1}{2}\left[-\begin{pmatrix}7&4&5\\11&8&11\\4&6&10\end{pmatrix}+\begin{pmatrix}0&3&6\\3&6&9\\9&3&3\end{pmatrix}+\begin{pmatrix}8&0&0\\0&8&0\\0&0&8\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}1&-1&1\\-8&6&-2\\5&-3&1\end{pmatrix}\end{aligned}$$

∎

Note: In this case, we were required to use this method to find the inverse. An alternate method of finding the inverse by performing elementary row and column operations is shown in 1985 question 1(c).

Question 3(b) *Find the eigenvalues and eigenvectors of*

$$\mathbf{A} = \begin{pmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{pmatrix}$$

Solution.

$$\begin{aligned}|\mathbf{A}-x\mathbf{I}| &= \begin{vmatrix}8-x&-6&2\\-6&7-x&-4\\2&-4&3-x\end{vmatrix} = 0\\ &\Rightarrow (8-x)(x^2-10x+21-16)+6(6x-18+8)+2(24-14+2x)=0\\ &\Rightarrow -x^3+18x^2-85x+40+36x-60+20+4x=0\\ &\Rightarrow x^3-18x^2+45x=0\end{aligned}$$

Thus the eigenvalues are $0, 3, 15$.

If (x_1, x_2, x_3) is an eigenvector for the eigenvalue 0, then $\begin{pmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \mathbf{0}$.

Thus $8x_1-6x_2+2x_3=0, -6x_1+7x_2-4x_3=0, 2x_1-4x_2+3x_3=0 \Rightarrow x_1=\frac{1}{2}x_3, x_2=x_3$. Thus $(1,2,2)$ is an eigenvector for 0, in general $(x/2, x, x), x \neq 0$ is an eigenvector for 0.

If (x_1, x_2, x_3) is an eigenvector for the eigenvalue 3, then $\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$. Thus $5x_1 - 6x_2 + 2x_3 = 0, -6x_1 + 4x_2 - 4x_3 = 0, 2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2, x_3 = -2x_2$. Thus $(2, 1, -2)$ is an eigenvector for 3, in general $(2x, x, -2x), x \neq 0$ is an eigenvector for 3.

If (x_1, x_2, x_3) is an eigenvector for the eigenvalue 15, then $\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$. Thus $-7x_1 - 6x_2 + 2x_3 = 0, -6x_1 - 8x_2 - 4x_3 = 0, 2x_1 - 4x_2 - 12x_3 = 0 \Rightarrow x_1 = 2x_3, x_2 = -2x_3$. Thus $(2, -2, 1)$ is an eigenvector for 15, in general $(2x, -2x, x), x \neq 0$ is an eigenvector for 15. ∎

Question 3(c) *Show that the eigenvalues of an upper or lower triangular matrix are just the diagonal elements of the matrix.*

Solution. Let $\mathbf{A} = (a_{ij})$, such that $a_{ij} = 0$ for $i < j$, i.e. $\mathbf{A}$ is upper triangular. Now

$$|x\mathbf{I} - \mathbf{A}| = (x - a_{11})(x - a_{22}) \ldots (x - a_{nn})$$

showing that $|x\mathbf{I} - \mathbf{A}| = 0 \Rightarrow x = a_{11}, a_{22}, \ldots, a_{nn}$. Thus the eigenvalues of $\mathbf{A}$ are $a_{11}, a_{22}, \ldots, a_{nn}$.

Similarly for a lower triangular matrix. ∎

Paper II

Question 4(a) *Prove that a necessary and sufficient condition that a linear transformation* $\mathbf{A}$ *on a unitary space is Hermitian is that* $\langle \mathbf{Ax}, \mathbf{x} \rangle$ *is real for all* $\mathbf{x}$.

Solution. A unitary space is an old name for an inner product space. Let $\mathcal{V}$ be an inner product space over $\mathbb{C}$, and $\langle \mathbf{Av}, \mathbf{v} \rangle$ be real for all $\mathbf{v} \in \mathcal{V}$. Then since

$$\langle \mathbf{A}(\mathbf{v} + \mathbf{w}), \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{Av}, \mathbf{v} \rangle + \langle \mathbf{Aw}, \mathbf{w} \rangle + \langle \mathbf{Av}, \mathbf{w} \rangle + \langle \mathbf{Aw}, \mathbf{v} \rangle$$

$\langle \mathbf{Av}, \mathbf{w} \rangle + \langle \mathbf{Aw}, \mathbf{v} \rangle$ is real (because $\langle \mathbf{A}(\mathbf{v} + \mathbf{w}), \mathbf{v} + \mathbf{w} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle - \langle \mathbf{Aw}, \mathbf{w} \rangle$ is real). Hence

$$\langle \mathbf{Av}, \mathbf{w} \rangle + \langle \mathbf{Aw}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{Av} \rangle + \langle \mathbf{v}, \mathbf{Aw} \rangle \qquad (1)$$

because z real $\Rightarrow z = \overline{z}$.

Also,

$$\langle \mathbf{A}(\mathbf{v} + i\mathbf{w}), \mathbf{v} + i\mathbf{w} \rangle = \langle \mathbf{Av}, \mathbf{v} \rangle + \langle \mathbf{A}(i\mathbf{w}), i\mathbf{w} \rangle - i\langle \mathbf{Av}, \mathbf{w} \rangle + i\langle \mathbf{Aw}, \mathbf{v} \rangle$$

thus $-i\langle \mathbf{Av}, \mathbf{w} \rangle + i\langle \mathbf{Aw}, \mathbf{v} \rangle$ is real. Thus

$$-i\langle \mathbf{Av}, \mathbf{w} \rangle + i\langle \mathbf{Aw}, \mathbf{v} \rangle = -i\langle \mathbf{w}, \mathbf{Av} \rangle + i\langle \mathbf{v}, \mathbf{Aw} \rangle \qquad (2)$$

Multiplying (1) by i and adding to (2), we get

$$2i\langle \mathbf{Aw}, \mathbf{v} \rangle = 2i\langle \mathbf{v}, \mathbf{Aw} \rangle$$

Thus $\mathbf{A} = \mathbf{A}^*$, so $\mathbf{A}$ is Hermitian.

Conversely, if $\langle \mathbf{Aw}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{Aw} \rangle$, then $\langle \mathbf{Av}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{Av} \rangle = \overline{\langle \mathbf{Av}, \mathbf{v} \rangle} \Rightarrow \langle \mathbf{Av}, \mathbf{v} \rangle$ is real. ∎

Question 4(b) *If* $\mathbf{A}$ *is a linear transformation on an* n*-dimensional vector space, then prove that*

1. $\operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}'$.
2. $\operatorname{nullity}\mathbf{A} = n - \operatorname{rank}\mathbf{A}$.

Solution. We know that $\operatorname{rank}\mathbf{A} = r$ if $\mathbf{A}$ has a minor of order r different from 0, and all minors of order $> r$ are 0. Thus $\operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}'$.

For the second part, see 1998 question 3(a). ∎

Question 4(c) *Show that a real symmetric matrix* $\mathbf{A}$ *is positive definite if and only if there exists a real non-singular matrix* $\mathbf{P}$ *such that* $\mathbf{A} = \mathbf{PP}'$.

Solution. $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{PP}'\mathbf{x} =$ sum of squares > 0 (because $\mathbf{P}'\mathbf{x} \neq \mathbf{0}$ as $\mathbf{P}$ is non-singular).

Conversely: Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ be a basis of $\mathbb{R}^n$. We will use this to construct a new basis $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ which satisfies $\mathbf{e_i}\mathbf{A}\mathbf{e_j} = \delta_{ij}$, as follows:

$$\begin{aligned}
\mathbf{e_1} &= \frac{\mathbf{x_1}}{\sqrt{\mathbf{x_1}\mathbf{A}\mathbf{x_1}}} \\
\mathbf{y_2} &= \mathbf{x_2} - (\mathbf{x_2}\mathbf{A}\mathbf{e_1})\mathbf{e_1} \\
\mathbf{e_2} &= \frac{\mathbf{y_2}}{\sqrt{\mathbf{y_2}\mathbf{A}\mathbf{y_2}}} \\
&\ldots \\
\mathbf{y_i} &= \mathbf{x_i} - \sum_{j=1}^{i-1}(\mathbf{x_i}\mathbf{A}\mathbf{e_j})\mathbf{e_j} \\
\mathbf{e_i} &= \frac{\mathbf{y_i}}{\sqrt{\mathbf{y_i}\mathbf{A}\mathbf{y_i}}} \\
&\ldots
\end{aligned}$$

$\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ are linearly independent — if $\sum_{i=1}^n a_i\mathbf{e_i} = \mathbf{0}$, then take the largest i such that $a_i \neq 0$, this allows us to express $\mathbf{x_i}$ in terms of the other basis vectors, which is not possible. Inductively we can also verify that $\mathbf{e_i}\mathbf{A}\mathbf{e_i} = 1$, and $\mathbf{e_i}\mathbf{A}\mathbf{e_j} = 0$ if $i \neq j$. This is the Gram-Schmidt orthonormalization process - we exploit the property that any positive definite matrix $\mathbf{A}$ gives rise to an inner product $\langle\mathbf{x}, \mathbf{y}\rangle = \mathbf{x}'\mathbf{A}\mathbf{y}$.

Now consider the matrix $\mathbf{Q} = \begin{pmatrix}\mathbf{e_1} & \mathbf{e_2} & \ldots & \mathbf{e_n}\end{pmatrix}$. Now if $\mathbf{B} = \mathbf{Q}'\mathbf{A}\mathbf{Q}$ then $b_{ij} = \mathbf{e_i}\mathbf{A}\mathbf{e_j}$, thus $\mathbf{B} = \mathbf{I}_n$. Since $\mathbf{Q}$ consists of linearly independent columns, it is invertible, and thus $\mathbf{A} = \mathbf{Q}'^{-1}\mathbf{Q}^{-1}$. Setting $\mathbf{P} = \mathbf{Q}'^{-1}$, we have $\mathbf{A} = \mathbf{PP}'$. ∎

Question 5(a) *If* $\mathbf{S}$ *is a skew symmetric matrix of order* n *and if* $\mathbf{I} + \mathbf{S}$ *is non-singular, then prove that* $\mathbf{A} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}$ *is an orthogonal matrix of order* n.

Solution. See 1999, question 2(b). ∎

Question 5(b) *Under what circumstances will the real $n \times n$ matrix*

$$\mathbf{A} = \begin{pmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ & & & \dots & \\ a & a & a & \dots & x \end{pmatrix}$$

be (1) positive semidefinite (2) positive definite.

Solution. The eigenvalues of the given matrix can be computed as follows:

$$\begin{vmatrix} x-\lambda & a & a & \dots & a \\ a & x-\lambda & a & \dots & a \\ a & a & x-\lambda & \dots & a \\ & & & \dots & \\ a & a & a & \dots & x-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-\lambda & a-x+\lambda & a-x+\lambda & \dots & a-x+\lambda \\ a & x-\lambda-a & 0 & \dots & 0 \\ a & 0 & x-\lambda-a & \dots & 0 \\ & & & \dots & \\ a & 0 & 0 & \dots & x-\lambda-a \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-\lambda+(n-1)a & 0 & 0 & \dots & 0 \\ a & x-\lambda-a & 0 & \dots & 0 \\ a & 0 & x-\lambda-a & \dots & 0 \\ & & & \dots & \\ a & 0 & 0 & \dots & x-\lambda-a \end{vmatrix} = 0$$

$$\Rightarrow (x-\lambda+(n-1)a)(x-\lambda-a)^{n-1} = 0$$

Thus the eigenvalues are $x-a$ (repeated $n-1$ times) and $x+(n-1)a$. For positive definite, $\lambda > 0 \Rightarrow x > a, x > (n-1)(-a)$. If $a > 0$, this reduces to $x > a$, if $a \leq 0$, this reduces to $x > (n-1)(-a)$.

For positive semi-definite, $\lambda \geq 0$. By the same reasoning, if $a > 0$, then $x \geq a$, otherwise $x \geq (n-1)(-a)$. ∎