UPSC Civil Services Main 1979 - Mathematics
Linear Algebra

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Mathura

Question 1(a) *State and prove Cayley-Hamilton Theorem.*

Solution. See 1987, question 5(a). ∎

Question 1(b) *Reduce the quadratic expression $x^2+2y^2+2z^2+2xy+2xz$ to the canonical form.*

Solution. Completing the squares: given form $= (x+y+z)^2+(y-z)^2$. Put $X = x+y+z, Y = y-z, Z = z$ to get the canonical form $= X^2+Y^2$. The expression is positive semi-definite.

Alternate solution: See 1981 question 1(b) for an alternate method of canonicalization. ∎

Question 2(a) *Find the elements p, q, r such that the product $\mathbf{BA}$ of the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \\ -10 & 2 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ q & r & 1 \end{pmatrix}$$

is of the form

$$\mathbf{BA} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}$$

Hence solve the set of equations $\mathbf{Ax} = \mathbf{y}$, where $\mathbf{x}$ is the column vector (x_1, x_2, x_3), and $\mathbf{y}$ is the column vector $(0, 8, -4)$.

Solution.

$$\mathbf{BA} = \begin{pmatrix} 1 & 2 & 1 \\ p+4 & 2p+1 & p+2 \\ q+4r-10 & 2q+r+2 & q+2r+4 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}$$

Thus $p+4=0, q+4r-10=0, 2q+r+2=0 \Rightarrow p=-4, r=\frac{22}{7}, q=-\frac{18}{7}$.

Now solving $\mathbf{Ax}=\mathbf{y}$ is the same as solving $\mathbf{BAx}=\mathbf{By}$ because $|\mathbf{B}| \neq 0$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -2 \\ 0 & 0 & \frac{54}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -\frac{18}{7} & \frac{22}{7} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ \frac{148}{7} \end{pmatrix}$$

Thus $\frac{54}{7}x_3 = \frac{148}{7} \Rightarrow x_3 = \frac{74}{27}$. $-7x_2 - 2x_3 = 8 \Rightarrow -7x_2 = 2x_3 + 8 = \frac{364}{27}$, so $x_2 = -\frac{52}{27}$. $x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = \frac{104}{27} - \frac{74}{27} = \frac{10}{9}$.

Thus $x_1 = \frac{10}{9}, x_2 = -\frac{52}{27}, x_3 = \frac{74}{27}$ is the required solution. ∎

Question 3(a) *If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a finite dimensional vector space, then show that*

$$\dim(\mathcal{S}+\mathcal{T}) = \dim\mathcal{S} + \dim\mathcal{T} - \dim(\mathcal{S}\cap\mathcal{T})$$

Solution. See 1988, question 1(b). ∎

Question 3(b) *Determine the value of a for which the following system of equations:*

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + 2x_2 + x_3 &= -2 \\ x_1 + x_2 + (a-5)x_3 &= a \end{aligned}$$

has (1) a unique solution (2) no solution.

Solution. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & a-5 \end{vmatrix} = 2a - 10 - 3 - a + 5 + 3 - 1 = a - 6$

1. If $a - 6 \neq 0$ i.e. $a \neq 6$, the system has a unique solution.
2. If $a = 6$, the system is inconsistent as the third equation becomes $x_1 + x_2 + x_3 = 6$, which is inconsistent with the first. So there is no solution.

∎

Paper II

Question 4(a) *Prove that any two finite dimensional vector spaces of the same dimension are isomorphic.*

Solution. See 1987 question 4(b). ∎

Question 4(b) *Define the dual space of a finite dimensional vector space $\mathcal{V}$ and show that it has the same dimension as $\mathcal{V}$.*

Solution. Let $\mathcal{V}^* = \{f : \mathcal{V} \longrightarrow \mathbb{R}, f \text{ a linear transformation}\}$. Then $\mathcal{V}^*$ is a vector space for the usual pointwise addition and scalar multiplication of functions: for all $\mathbf{v} \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$, $(f+g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v}), (\alpha f)(\mathbf{v}) = \alpha f(\mathbf{v})$.

Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be a basis for $\mathcal{V}$. Define n linear functionals $v_1^*, \ldots, v_n^*$ by $v_i^*(\mathbf{v_j}) = \delta_{ij}$, and $v_i^*(\sum_{j=1}^n \alpha_j \mathbf{v_j}) = \sum_{j=1}^n \alpha_j v_i^*(\mathbf{v_j}) = \alpha_i$.

Then $v_1^*, \ldots, v_n^*$ are linearly independent — $\sum_{i=1}^n \alpha_i v_i^* = 0 \Rightarrow (\sum_{i=1}^n \alpha_i v_i^*)(\mathbf{v_j}) = \alpha_j = 0,\ 1 \leq j \leq n$.

$v_1^*, \ldots, v_n^*$ generate $\mathcal{V}^*$ — if $f \in \mathcal{V}^*$, then $f = \sum_{i=1}^n f(\mathbf{v_i}) v_i^*$. Clearly $(\sum_{i=1}^n f(\mathbf{v_i}) v_i^*)(\mathbf{v_j}) = \sum_{i=1}^n f(\mathbf{v_i}) v_i^*(\mathbf{v_j}) = f(\mathbf{v_j})$, so the two sides agree on $\mathbf{v_1}, \ldots, \mathbf{v_n}$, and hence by linearity on all of $\mathcal{V}$.

Thus $v_1^*, \ldots, v_n^*$ is a basis of $\mathcal{V}^*$, so $\dim \mathcal{V}^* = \dim \mathcal{V}$. $\mathcal{V}^*$ is called the dual of $\mathcal{V}$. ∎

Question 4(c) *Show that every finite dimensional inner product space $\mathcal{V}$ over the field of complex numbers has an orthonormal basis.*

Solution. Let $\mathbf{w_1}, \ldots, \mathbf{w_n}$ be a basis of $\mathcal{V}$. We will convert it into an orthonormal basis of $\mathcal{V}$ by the Gram-Schmidt orthonormalization process.

Starting with $i = 1$, define

$$\mathbf{v_i} = \mathbf{w_i} - \sum_{j=1}^{i-1} \frac{\langle \mathbf{w_i}, \mathbf{v_j} \rangle}{||\mathbf{v_j}||^2} \mathbf{v_j}$$

Each $\mathbf{v_i}$ is non-zero, as otherwise $\mathbf{w_i}$ can be written as a linear combination of $\mathbf{w_j}, j < i$, but $\mathbf{w_1}, \ldots, \mathbf{w_n}$ are linearly independent.

Now we can prove by induction on i that $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0$ for all $j < i$ — this is enough because $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \overline{\langle \mathbf{v_j}, \mathbf{v_i} \rangle}$. Suppose it is true for all $k < i$. Then $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \langle \mathbf{w_i}, \mathbf{v_j} \rangle - \sum_{m=1}^{i-1} \frac{\langle \mathbf{w_i}, \mathbf{v_m} \rangle}{||\mathbf{v_m}||^2} \langle \mathbf{v_m}, \mathbf{v_j} \rangle = \langle \mathbf{w_i}, \mathbf{v_j} \rangle - \frac{\langle \mathbf{w_i}, \mathbf{v_j} \rangle}{||\mathbf{v_j}||^2} \langle \mathbf{v_j}, \mathbf{v_j} \rangle = 0$. Thus $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are mutually orthogonal. They are linearly independent, as $\sum_{i=1}^n a_i \mathbf{v_i} = \mathbf{0} \Rightarrow \langle \sum_{i=1}^n a_i \mathbf{v_i}, \mathbf{v_j} \rangle = a_j ||\mathbf{v_j}||^2 = 0 \Rightarrow a_j = 0$ for all $i \leq j \leq n$. Replacing $\mathbf{v_i}$ by $\frac{\mathbf{v_i}}{||\mathbf{v_i}||}$ gives us an orthonormal basis of $\mathcal{V}$. ∎

Question 5(a) *Define the rank and nullity of a linear transformation. If $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional vector spaces over a field, and T is a linear transformation of $\mathcal{V}$ into $\mathcal{W}$, prove that*

$$\operatorname{rank} T + \operatorname{nullity} T = \dim \mathcal{V}$$

Solution. See 1998 question 3(a). ∎

Question 5(b) *Define a positive definite form. State and prove a necessary and sufficient condition for a quadratic form to be positive definite.*

Solution. See 1992 question 2(c). ∎

Question 5(c) *Show that the mapping* $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *defined by* $T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z)$ *is a linear transformation. Find its nullity.*

Solution.

$$\begin{aligned}
T(a\mathbf{x} + b\mathbf{y}) &= T(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \\
&= (ax_1 + by_1 - ax_2 - by_2 + 2ax_3 + 2by_3, 2ax_1 + 2by_1 + ax_2 + by_2, \\
&\quad\ -ax_1 - by_1 - 2ax_2 - 2by_2 + 2ax_3 + 2by_3) \\
&= aT(x_1, x_2, x_3) + bT(y_1, y_2, y_3)
\end{aligned}$$

Thus T is a linear transformation.

If $(x, y, z) \in$ the null space of T, then $x - y + 2z = 0, 2x + y = 0, -x - 2y + 2z = 0 \Rightarrow y = -2x, z = -\frac{3x}{2}$. Thus the null space is $\{(x, -2x, -\frac{3x}{2}) \mid x \in \mathbb{R}\} = \{(2, -4, -3)x \mid x \in \mathbb{R}\}$.

Thus nullity $T = 1$. ∎